CSE 311 : Problems for Midterm Review Solutions

This is a large set of problems intended to cover many of the topics we have seen so far this quarter. You should not view this as an "exam" (e.g. there are far too many problems here to do in a single sitting) but we recommend you treat individual problems in the style of an exam (so allowing yourself to look at resources as needed, but not looking at the solutions until after you've attempted the problem).

1. Logic

(a) Show that the expression $(p \to q) \to (p \to r)$ is a contingency. **Solution:**

Under the assignment $p = \mathsf{T}, q = \mathsf{T}, r = \mathsf{T}$, $(p \to q) \to (p \to r)$ evaluates to T. but under the assignment $p = \mathsf{T}, q = \mathsf{T}, r = \mathsf{F}$, it evaluates to F (since $p \to q$ evaluates to T and $p \to r$ evaluates to F). Therefore, it is a contingency.

(b) Give an expression that is logically equivalent to $(p \to q) \to (p \to r)$ using the logical operators $\neg$, $\vee$, and $\wedge$ (but not $\to$). **Solution:**

$\neg(\neg p \vee q) \vee (\neg p \vee r)$ and $(p \wedge \neg q) \vee \neg p \vee r$ are two natural choices here.

(c) Determine whether the following compound proposition is a tautology, a contradiction, or a contingency: $((s \vee p) \wedge (s \vee \neg p)) \to ((p \to q) \to r)$. **Solution:**

This is a contingency: Under the truth assignment $s = \mathsf{T}$, $p = \mathsf{F}$, $q = \mathsf{T}$ and $r = \mathsf{F}$, it evaluates to F because we have $((s \vee p) \wedge (s \vee \neg p)) = \mathsf{T}$ and $((p \to q) \to r) = \mathsf{F}$ because $(p \to q) = \mathsf{T}$ and $r = \mathsf{F}$. On the other hand if all of p, q, r, s are F, the whole formula evaluates to T.

(d) Show that the following is a tautology: $(((\neg p \vee q) \wedge (p \vee r)) \to (q \vee r))$. **Solution:**

Solution 1: Truth table:

p	q	r	$\neg p$	$\neg p \vee q$	$p \vee r$	$(\neg p \vee q) \wedge (p \vee r)$	$q \vee r$	$((\neg p \vee q) \wedge (p \vee r)) \to (q \vee r)$
F	F	F	T	T	F	F	F	T
F	F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
T	F	F	F	F	T	F	F	T
T	F	T	F	F	T	F	T	T
T	T	F	F	T	T	T	T	T
T	T	T	F	T	T	T	T	T

Solution 2: Derivation:

	1.1 $\neg(q \vee r)$	Assumption
	1.2 $\neg q \wedge \neg r$	De Morgan's Law from 1.1
	1.3 $p \vee \neg p$	Excluded Middle
	1.4 $(p \vee \neg p) \wedge (\neg q \wedge \neg r)$	Intro $\wedge$ from 1.3 and 1.2
	1.5 $(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$	Distributive Law from 1.4
	1.6 $((p \wedge \neg q) \vee (\neg p \wedge \neg q \wedge \neg r)) \wedge (\neg r \vee (\neg p \wedge \neg q \wedge \neg r))$	Associativity and Distributive Law from 1.5
	1.7 $(p \wedge \neg q) \vee (\neg p \wedge \neg q \wedge \neg r)$	Elim $\wedge$ from 1.6
	1.8 $((p \wedge \neg q) \vee (\neg p \wedge \neg r)) \wedge ((p \wedge \neg q) \vee \neg q)$	Commutativity and Distributive Law from 1.7
	1.9 $(p \wedge \neg q) \vee (\neg p \wedge \neg r)$	Elim $\wedge$ from 1.8
	1.10 $(\neg\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$	Double Negation from 1.9
	1.11 $\neg(\neg p \vee q) \vee \neg(p \vee r)$	De Morgan's Law (twice) from 1.10
	1.12 $\neg((\neg p \vee q) \wedge (p \vee r))$	De Morgan's Law from 1.11
2.	$\neg(q \vee r) \rightarrow \neg[(\neg p \vee q) \wedge (p \vee r)]$	Direct Proof Rule
3.	$[(\neg p \vee q) \wedge (p \vee r)] \rightarrow (q \vee r)$	Contrapositive (from 2)

2. Boolean Algebra

Write a boolean algebra expression equivalent to $(p \rightarrow q) \rightarrow r$ that is:

(a) A sum of products **Solution:**

$pq' + r$.

(b) A product of sums **Solution:**

$(p + r)(q' + r)$.

3. Predicate Logic

(a) Using the predicates:

$\mathsf{Likes}(p, f)$: "Person p likes to eat the food f."

$\mathsf{Serves}(r, f)$: "Restaurant r serves the food f."

$\mathsf{Restaurant}(r)$: "r is a restaurant."

$\mathsf{Food}(f)$: "f is a food."

$\mathsf{Person}(p)$: "p is a restaurant."

translate the following statements into logical expressions. In your translations, let your domain of discourse be "restaurants, food, people"

- Every restaurant serves a food that no one likes. **Solution:**

$\forall r \exists f(\mathsf{Restaurant}(r) \rightarrow [\mathsf{Serves}(r, f) \wedge \forall p[\mathsf{Person}(p) \rightarrow (\mathsf{Food}(f) \wedge \neg\mathsf{Likes}(p, f))]])$ or $\forall r \exists f \forall p(\mathsf{Restaurant}(r) \rightarrow \mathsf{Serves}(r, f) \wedge \neg\mathsf{Likes}(p, f) \wedge \mathsf{Food}(f))$.

Restricting the domains of r is required here (without it, plugging in a non-restaurant would cause the $\forall$ to evaluate to false). Because the predicates `Likes` and `Serves` require the inputs to be part of particular domains, further domain restriction turns out to be unneccessary in the second formulation (though it would not be incorrect to add). If in doubt, restrict the domain explicitly (just make sure you're doing it properly).

- Every restaurant that serves TOFU also serves a food which RANDY does not like. **Solution:**

$\forall r(\mathsf{Serves}(r, \text{TOFU}) \rightarrow \exists f(\mathsf{Serves}(r, f) \wedge \neg\mathsf{Likes}(\text{RANDY}, f))$ or
$\forall r \exists f(\mathsf{Serves}(r, \text{TOFU}) \rightarrow (\mathsf{Serves}(r, f) \wedge \neg\mathsf{Likes}(\text{RANDY}, f))$.

You can also add in domain restrictions, but they are not required in this part (the only way to get Serves and Likes to evaluate to true are for the variables to be in the right restriction already).

(b) Let $P(x, y)$ be the predicate "$x < y$" and let the universe for all variables be the real numbers. Express each of the following statements as predicate logic formulas using P:

- For every number there is a smaller one. **Solution:**

$\forall x \exists y P(y, x)$.

- 7 is smaller than any other number. **Solution:**

$\forall y((y \neq 7) \rightarrow P(7, y))$.

- 7 is between a and b. (Don't forget to handle both the possibility that b is smaller than a as well as the possibility that a is smaller than b.) **Solution:**

$(P(a, 7) \wedge P(7, b)) \vee (P(b, 7) \wedge P(7, a))$

- Between any two different numbers there is another number. **Solution:**

$\forall x \forall y((x \neq y) \rightarrow \exists z((P(x, z) \wedge P(z, y)) \vee (P(y, z) \wedge P(z, x))$ or
$\forall x \forall y \exists z((x \neq y) \rightarrow ((P(x, z) \wedge P(z, y)) \vee (P(y, z) \wedge P(z, x))$.

- For any two numbers, if they are different then one is less than the other. **Solution:**

$\forall x \forall y((x \neq y) \rightarrow (P(x, y) \vee P(y, x)))$.

(c) Let $V(x, y)$ be the predicate "x voted for y", let $M(x, y)$ be the predicate "x received more votes than y", and let the universe for all variables be the set of all people. Express each of the following statements as predicate logic formulas using V and M:

- Everybody received at least one vote. **Solution:**

$\forall x \exists y V(y, x)$.

- Jane and John voted for the same person. (you do not have to explicitly require that Jane and John did not cast more that one vote) **Solution:**

$\exists x(V(\text{Jane}, x) \wedge V(\text{John}, x))$.

- Ross won the election. (The winner is the person who received the most votes.) **Solution:**

$\forall x((x \neq \text{Ross}) \to M(\text{Ross}, x))$.

- Nobody who votes for him/herself can win the election. **Solution:**

Lots of good answers here; two possible answers: $\neg\exists x(V(x,x) \wedge \forall y((y \neq x) \to M(x,y)))$ or $\forall x(V(x,x) \to \exists y M(y,x))$.

- Everybody can vote for at most one person. **Solution:**

$\forall x \forall y \forall z((V(x,y) \wedge V(x,z)) \to (y = z))$ or $\forall x \forall y \forall z((y \neq z) \to (\neg V(x,y) \vee \neg V(x,z)))$.

(d) Find predicates $P(x)$ and $Q(x)$ and a domain of discourse such that $\forall x(P(x) \oplus Q(x))$ is true, but $\forall x P(x) \oplus \forall x Q(x)$ is false. **Solution:**

Let $P(x)$ be "x is even" and let $Q(x)$ be "x is odd" and let the universe be the set of all integers. Every integer is either even or odd but not both so $\forall x(P(x) \oplus Q(x))$ is true, but not all integers are even and not all integers are odd, so $\forall x P(x)$ and $\forall x Q(x)$ are both false and hence $\forall x P(x) \oplus \forall x Q(x)$ is false.

4. Formal Proofs

Use rules of inference to show that if the premises $\forall x(P(x) \to Q(x))$, $\forall x(Q(x) \to R(x))$, and $\neg R(i)$, where i is in the domain, are true, then the conclusion $\neg P(i)$ is true. (Note: You do not need to give the names for the rules of inference.) **Solution:**

1.$\forall x(P(x) \to Q(x))$	Given
2.$\forall x(Q(x) \to R(x))$	Given
3.$\neg R(i)$	Given
4.$Q(i) \to R(i)$	Elim $\forall$ from 2
5.$\neg R(i) \to \neg Q(i)$	Contrapositive from 4
6.$\neg Q(i)$	Modus Ponens from 3 and 5
7.$P(i) \to Q(i)$	Elim $\forall$ from 1
8.$\neg Q(i) \to \neg P(i)$	Contrapositive from 7
9. ¬P(i)	Modus Ponens from 6 and 8

5. English Proofs

(a) Prove that if n is even and m is odd, then $(n+1)(m+1)$ is even. **Solution:**

Suppose that n is even and m is odd.
Since m is odd there is some integer ℓ such that $m = 2\ell + 1$.
It follows that $m + 1 = 2\ell + 2 = 2(\ell + 1)$.
Therefore $(n+1)(m+1) = 2(n+1)(\ell+1)$.
Since n and ℓ are integers, $(n+1)(\ell+1)$ is an integer.
Therefore $(n+1)(m+1)$ is 2 times an integer $(n+1)(\ell+1)$ and therefore $(n+1)(m+1)$ is even.

(b) Prove or disprove:

- For positive integers x, p, and q, $(x\%p)\%q = x\%pq$. **Solution:**

  This is false. For a counterexample you can choose $p = 2$, $q = 3$ and $x = 3$. In this case $x\%p = 1$ and so $(x\%p)\%q = 1$. On the other hand $x\%pq = 3\%6 = 3$ so they are not equal.

- For positive integers x, p, and q, $(x\%p)\%q = (x\%q)\%p$. **Solution:**

  This is also false. We can take the same values $p = 2$, $q = 3$ and $x = 3$ from part (i). As we have seen, $(x\%p)\%q = 1$. On the other hand, $x\%q = 0$ so $(x\%q)\%p = 0$ so they are not equal.

(c) Prove that the sum of an odd number and an even number is an odd number. **Solution:**

Suppose that n is odd and m is even. Then there exist integers k and ℓ such that $n = 2k + 1$ and $m = 2\ell$. Therefore $n + m = 2k + 1 + 2\ell = 2(k + \ell) + 1$. Since $k + \ell$ is an integer, $n + m$ is 1 more than twice an integer and thus $n + m$ is odd.

6. Induction

(a) Prove the following for all natural numbers n by induction, $\displaystyle\sum_{i=0}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$. **Solution:**

Proof:

Let $P(n)$ be "$\sum_{i=0}^{n} \frac{i}{2^i} = 2 - (n+2)/2^n$". We will prove by induction that $P(n)$ is true for all $n \geq 0$.

Base Case: $\sum_{i=0}^{0} \frac{i}{2^i} = 0 \cdot 2^0 = 0$. On the other hand $2 - (0+2)/2^0 = 2 - 2/1 = 0$. Therefore $\sum_{i=0}^{0} \frac{i}{2^i} = 2 - (0+2)/2^0$ and thus $P(0)$ is true.

Inductive Hypothesis: Assume that $\sum_{i=0}^{k} \frac{i}{2^i} = 2 - (k+2)/2^k$ for some arbitrary integer $k \geq 0$.

Inductive Step: Now

$$\begin{aligned}
\sum_{i=0}^{k+1} \frac{i}{2^i} &= \sum_{i=0}^{k} \frac{i}{2^i} + (k+1)/2^{k+1} && \text{by definition} \\
&= 2 - (k+2)/2^k + (k+1)/2^{k+1} && \text{by the Inductive Hypothesis} \\
&= 2 - [2(k+2) - (k+1)]/2^{k+1} \\
&= 2 - (k+4-1)]/2^{k+1} \\
&= 2 - (k+3)/2^{k+1}
\end{aligned}$$

which is what we wanted to prove.

Therefore by the principle of induction we have shown that $\sum_{i=0}^{n} \frac{i}{2^i} = 2 - (n+2)/2^n$ for all $n \geq 0$.

(b) Let $T(n)$ be defined by: $T(0) = 1$, $T(n) = 2nT(n-1)$ for $n \geq 1$. Prove that for all $n \geq 0$, $T(n) = 2^n n!$. **Solution:**

Proof:

Let $P(n)$ be "$T(n) = 2^n n!$". We will prove by induction that $P(n)$ is true for all $n \geq 0$.

Base Case: $2^0 0! = 1 \cdot 1 = 1 = T(0)$. Therefore $P(0)$ is true.

Inductive Hypothesis: Assume that $T(k) = 2^k k!$ for some arbitrary integer $k \geq 0$.

Inductive Step:

$$\begin{array}{rcll} T(k+1) & = & 2(k+1)T(k) & \text{by definition since } k+1 \geq 1 \\ & = & 2(k+1)2^k k! & \text{by the Inductive Hypothesis} \\ & = & 2^{k+1}(k+1)k! & \\ & = & 2^{k+1}(k+1)! & \text{by definition of factorial} \end{array}$$

which is what we wanted to prove.

Therefore by the principle of induction we have shown that $T(n) = 2^n n!$ for all $n \geq 0$.

(c) Let $x_1, x_2, \ldots, x_n$ be odd integers. Prove by induction that $x_1 x_2 \cdots x_n$ is also an odd integer. **Solution:**

Proof:

Let $P(n)$ be "$x_1 x_2 \cdots x_n$ is an odd integer". We will prove by induction that $P(n)$ is true for all $n \geq 1$.

Base Case: Since x_1 is an odd integer, the product (which is just x_1) is odd. Therefore $P(1)$ is true.

Inductive Hypothesis: Assume that $x_1 x_2 \cdots x_k$ is an odd integer for some arbitrary integer $k \geq 1$.

Inductive Step:

By the Inductive Hypothesis $x_1 x_2 \cdots x_k$ is an odd integer so there is some integer ℓ such that $x_1 x_2 \cdots x_k = 2\ell + 1$. Since x_{k+1} is an odd integer there is some integer m such that $x_{k+1} = 2m + 1$. Therefore

$$x_1 x_2 \cdots x_{k+1} = x_1 x_2 \cdots x_k \cdot x_{k+1} = (2\ell + 1)(2m + 1) = 4\ell m + 2\ell + 2m + 1 = 2(2\ell m + \ell + m) + 1.$$

Since $(2\ell m + \ell + m)$ is an integer, $x_1 x_2 \cdots x_{k+1}$ is an odd integer, which is what we wanted to prove.

Therefore by the principle of induction we have shown that $x_1 x_2 \cdots x_n$ is an odd integer for all $n \geq 1$.

(d) Use mathematical induction to show that 3 divides $n^3 - n$ whenever n is a non-negative integer. **Solution:**

Proof:
Let $P(n)$ be "3 divides $n^3 - n$". We will prove by induction that $P(n)$ is true for all $n \geq 0$.

Base Case: $0^3 - 0 = 0 = 3 \cdot 0$ therefore 3 divides $0^3 - 0$ so $P(0)$ is true.

Inductive Hypothesis: Assume that 3 divides $k^3 - k$ for some arbitrary integer $k \geq 0$.

Inductive Step:

Since by the Inductive Hypothesis 3 divides $k^3 - k$, there is some integer ℓ such that $k^3 - k = 3\ell$. Now

$$\begin{array}{rcl} (k+1)^3 - (k+1) & = & k^3 + 3k^2 + 3k + 1 - (k+1) \\ & = & k^3 + 3k^2 + 3k - k \\ & = & 3\ell + 3k^2 + 3k \\ & = & 3(\ell + k^2 + k) \end{array}$$

Since $\ell + k^2 + k$ is an integer, we have shown that 3 divides $(k+1)^3 - (k+1)$ which is what we wanted to prove.

Therefore by the principle of induction we have shown that 3 divides $n^3 - n$ for all $n \geq 0$.

7. Euclidean Algorithm

(a) Use Euclid's algorithm to help you solve $11x \equiv 4(\text{mod} 27)$ for x. **Solution:**

We run Euclid's algorithm to compute gcd$(27, 11)$.

$$\begin{aligned} 27 &= 2 \cdot 11 + 5 \\ 11 &= 2 \cdot 5 + 1 \\ 5 &= 5 \cdot 1 + 0 \end{aligned}$$

Therefore $1 = 11 - 2 \cdot 5 = 11 - 2(27 - 2 * 11) = (-2) \cdot 27 + 5 \cdot 11$. Therefore 5 is the multiplicative inverse of 11 modulo 27. It follows that $x = 5 \cdot 4 = 20$ solves $11x \equiv 4(\text{mod} 27)$. (We can check that 27 times 8 is 216 and 11 times 20 is 220.)

(b) Find the multiplicative inverse of 2 modulo 9 (in other words, find a solution to the equation $2x$ mod $9 = 1$.) **Solution:**

We run Euclid's algorithm to compute gcd$(9, 2)$ which is 1:The first step is $9 = 4 \cdot 2 + 1$ and of course we are done. Therefore $1 = 1 \cdot 9 - 4 \cdot 2$. The multiplicative inverse of 2 is then (-4) mod $9 = 5$. This is so easy you could do it by trying all possibilities.

(c) Which integers in $\{1, 2, \ldots, 8\}$ have multiplicative inverses modulo 9? **Solution:**

This is which integers x in have gcd$(x, 9) = 1$ so it is: $\{1, 2, 4, 5, 7, 8\}$.

8. Sets

Let your domain of discourse be integers (equivalently, your universal set $\mathcal{U} = \mathbb{Z}$). Define $S = \{x : x \equiv 2 \pmod 5\}$ and $T = \{x : 2x \equiv 4 \pmod 5\}$

(a) Show $S \subseteq T$ **Solution:**

Let x be an arbitrary element of S. By definition $x \equiv 2 \pmod 5$. Multiplying the equivalance by 2, we have $2x \equiv 4 \pmod 5$. Thus $x \in T$ by definition of T. Since an arbitrary element of S is also in T, we have $S \subseteq T$.

(b) Show $T \subseteq S$. You may use the fact that 3 is the multiplicative inverse of $2 \pmod 5$. **Solution:**

Let x be an arbitrary element of T. By definition of T, $2x \equiv 4 \pmod 5$. We have:

$$\begin{aligned} 2x &\equiv 4 \pmod 5 && \text{From definition of } S \\ 3 \cdot 2x &\equiv 3 \cdot 4 \pmod 5 && \text{Multiplying by } 3 \\ x &\equiv 12 \pmod 5 && 2 \text{ is an inverse of } 3 \\ x &\equiv 2 \pmod 5 && 12\%5 = 2 \text{ so } 2 \equiv 12 \pmod 5 \end{aligned}$$

Thus $x \in S$ as well, and we have $T \subseteq S$.

Let $S' = \{x : x \equiv 3 \pmod{10}\}$ and $T' = \{x : 2x \equiv 6 \pmod{10}\}$

(a) Show that $S' \neq T'$. **Solution:**

Observe that $8 \in T'$ Because $2x = 16$ and $16 - 6 = 10 \cdot 1$, so $10 | 2x - 6$ and thus $2x \equiv 6 \pmod{10}$. But $8 \notin S'$ because $3\%10 = 3 \neq 8 = 8\%10$, these are equal iff $8 \equiv 3 \pmod{10}$.

(b) One of the following is true: $S' \subseteq T'$, $T' \subseteq S'$. Determine which is true, and prove it. **Solution:**

$S' \subseteq T'$ is true. Let x be an arbitrary element of S'. By definition of S', $x \equiv 3 \pmod{10}$. Multiplying the equivalence by 2, we have $2x \equiv 6 \pmod{10}$. So by definition, $x \in T'$, as required.