CSE 311 : Practice Midterm Solutions

This exam is a (slight) modification of a real midterm given in a prior quarter of CSE311.

The original exam was given in a 50 minute slot; you should expect to spend more time on the Fall 2020 exam (both because you'll be able to write or type more careful solutions than under time pressure and because I'm comfortable giving questions I expect to take slightly longer (since we're not under as much time pressure).

We strongly recommend you take this exam as though it were closed book – even though your exam will be open book.

Instructions

- Students had 50 minutes to complete the exam.
- The exam was closed resource (except for the logical equivalences, boolean algebra, and inference rules reference sheets). You exam will be open resource.
- The problems are of varying difficulty.
- If you get stuck on a problem, move on and come back to it later.

Topic	**Max points**
1. Translation	20
2. Induction	15
3. Inverses	10
4. Circuits	20
5. [Ir]rational I	5
6. [Ir]rational II	10
7. Squares and mod	20
Total	**100**

1. To Logic...or Not To Logic

1.1. Choose your own predicate adventure [10 points]

(a) Choose a meaning of $P(x, y, z)$ such that $\forall x\ \exists y\ \forall z\ P(x, y, z)$ is false, but $\forall x\ \forall y\ \exists z\ P(x, y, z)$ is true.

Solution:

Let the domain be $\mathbb{N}$. Let $P(x, y, z)$ be "$x \geq 0 \wedge y \geq z$".

Then, the first statment is false, because, while $x \geq 0$ for everything in the domain, there is no largest number in the domain. However, the second statement is true, because $x \geq 0$ and $z = y$ satisfies the second part.

(b) In the domain of integers, using any standard mathematical notation (but no new predicates), define $\texttt{Prime}(x)$ to mean "x is prime".

Solution:

$\texttt{Prime}(x) \equiv x \geq 2 \wedge \forall y\ ((1 \leq y \leq x \wedge y \mid x) \rightarrow (y = x \vee y = 1))$

1.2. Games [10 points]

Let the predicates $D(x, y)$ mean "team x defeated team y" and $P(x, y)$ mean "team x has played team y." Give quantified formulas with the following meanings:

(a) Every team has lost at least one game.

Solution:

$\forall x\ \exists y\ D(y, x)$

(b) There is a team that has beaten every team it has played.

Solution:

$\exists x\ \forall y\ (P(x, y) \rightarrow D(x, y))$

2. Obvious Induction Problem [15 points]

Prove for all $n \in \mathbb{N}$ that the following identity is true:

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

where $x \in \mathbb{R}, x \neq 1$.

Solution:

Let $P(n)$ be the statement "$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$"

We show for an arbitrary $x \neq 1$, $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: When $n = 0$, Since $x \neq 1$,

$$\sum_{i=0}^{0} x^i = x^0 = 1 = \frac{1 - x^1}{1 - x}$$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \in \mathbb{N}$, $k \geq 0$.

Inductive Step: We see that

$$\begin{aligned}
\sum_{i=0}^{k+1} x^i &= \sum_{i=0}^{k} x^i + x^{k+1} && \text{[Taking out the last term]} \\
&= \frac{1 - x^{k+1}}{1 - x} + x^{k+1} && \text{[By the IH]} \\
&= \frac{(1 - x^{k+1}) + (1 - x)x^{k+1}}{1 - x} && \text{[Algebra]} \\
&= \frac{1 - x^{k+2}}{1 - x} && \text{[Simplifying]}
\end{aligned}$$

which is $P(k + 1)$.

Thus, we have proven $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

3. 311 is Prime! [10 points]

Find all solutions in the range $0 \leq x < 311$ to the modular equation:

$$12x \equiv 5 \pmod{311}$$

Solution:

First, we compute the gcd of 311 and 12.

$$\begin{aligned} 311 &= 25 \cdot 12 + 11 \\ 12 &= 1 \cdot 11 + 1 \\ 11 &= 11 \cdot 1 + 0 \end{aligned}$$

so $gcd(311, 12) = 1$ and hence we finish the Extended Euclidean Algorithm using:

$$\begin{aligned} 11 &= 311 - 25 \cdot 12 \\ 1 &= 12 - 11 \cdot 1 \end{aligned}$$

Now, backwards substituting:

$$1 = 12 - 1 \cdot 11 = 12 - 1 \cdot (311 - 25 \cdot 12) = 26 \cdot 12 - 1 \cdot 311$$

So, the multiplicative inverse of 12 modulo 311 is 26.

Now, we have the modular equation $12(26) \equiv 1 \pmod{311}$. Multiplying both sides by 5, we get:

$$12(26 \cdot 5) \equiv 5 \pmod{311} \rightarrow 12(130) \equiv 5 \pmod{311}$$

So, $x = 130$.

4. Even Circuits Are Fun [20 points]

The function `multiple-of-three` takes in two inputs: $(x_1x_0)_2$ and outputs 1 iff $3 \mid (x_1x_0)_2$.

(a) Draw a table of values (e.g. a truth table) for `multiple-of-three`.

Solution:

x_1	x_0	`multiple-of-three`
0	0	1
0	1	0
1	0	0
1	1	1

(b) Write `multiple-of-three` as a sum-of-products.

Solution:

$$\texttt{multiple-of-three} = (x_1'x_0') + (x_1x_0)$$

(c) Write `multiple-of-three` as a product-of-sums.

Solution:

$$\texttt{multiple-of-three} = (x_1 + x_0')(x_1' + x_0)$$

(d) Write `multiple-of-three` as a simplified expression (don't bother explaining what rules you're using).

Solution:

$$\texttt{multiple-of-three} = (x_1 + x_0)' + (x_1x_0)$$

5. Irrationally Rational [5 points]

Recall the definition of irrational is that a number is not rational, and that

$$\texttt{Rational}(x) \equiv \exists p\ \exists q\ x = \frac{p}{q} \wedge \texttt{Integer}(p) \wedge \texttt{Integer}(q) \wedge q \neq 0$$

For this question, you may assume that π is irrational. Disprove that if x and y are irrational, then x + y is irrational.

Solution:

Note that π is irrational, and multiplying by -1 maintains irrationality (because if it didn't, then we could find p, q by multiplying by -1, getting p, q, and choosing $-p$ and q). Finally, note that $\pi + (-\pi) = 0$, which is rational.

6. Rationally Irrational [10 points]

Recall the definition of irrational is that a number is not rational, and that

$$\texttt{Rational}(x) \equiv \exists p\ \exists q\ x = \frac{p}{q} \land \texttt{Integer}(p) \land \texttt{Integer}(q) \land q \neq 0$$

Prove that if x and y are rational and $x \neq 7$, then $\frac{y^2}{x-7}$ is rational.

Solution:

Suppose x, y are rational and $x \neq 7$. Then, by definition $x = p/q$ for some integers p and q with $q \neq 0$. Therefore $x - 7 = p/q - 7 = \frac{p-7q}{q}$. Since $x \neq 7$, we have $p - 7q \neq 0$. It follows that $\frac{1}{x-7} = \frac{q}{p-7q}$ is rational since q and $p - 7q$ are integers and $p - 7q \neq 0$. Now since $\frac{y^2}{x-7} = y \cdot y \cdot \frac{1}{x-7}$ we see that $\frac{y^2}{x-7}$ is the product of three rational numbers. In class, we showed that the product of rational numbers is also rational. Since y and $\frac{1}{x-7}$ are both rational, the product $y \cdot y \cdot \frac{1}{x-7} = \frac{y^2}{x-7}$ is also rational as required.

7. Gotta $\square$ m $\forall$ [20 points]

We say that k is *a square modulo* m iff there is some integer j such that $k \equiv j^2 \pmod{m}$.
Let $T = \{m \ : \ m = n^2 + 1 \text{ for some integer } n\}$.

(a) Prove that if $m \in T$, then -1 is a square modulo m. (8 points)

Solution:

Let m be an arbitrary element of T.

Then, $m = n^2 + 1$ for some integer n by definition of T.

Therefore, $m \mid (n^2 + 1)$. So, $n^2 \equiv -1 \pmod{m}$, which means -1 is a square modulo m.

(b) Prove that for all integers m and k, if $m \in T$ and k is a square modulo m then $-k$ is also a square modulo m. (12 points)

Solution:

Let m be an arbitrary element of T, and suppose that k is a square modulo m. Then, $k \equiv j^2 \pmod{m}$ for some integer j.

Multiplying both sides of the congruence by -1, we get $-k \equiv (-1)j^2 \pmod{m}$.

From (a), we know that $n^2 \equiv -1 \pmod{m}$. Thus, we have $-k \equiv n^2 j^2 \pmod{m}$.

So, $-k \equiv (nj)^2 \pmod{m}$, which means $-k$ is a square modulo m.