CSE 311 : Practice Final Solutions

This exam is a (slight) modification of a real final given in a prior quarter of CSE311.

The original exam was given in a 110 minute slot.

We strongly recommend you take this exam as though it were closed book – even though your exam will be open book.

Instructions

- Students had 110 minutes to complete the exam.
- The exam was closed resource (except for the logical equivalences, boolean algebra, and inference rules reference sheets). You exam will be open resource.
- The problems are of varying difficulty.
- If you get stuck on a problem, move on and come back to it later.

1. Regularly Irregular [15 points]

Let $\Sigma = \{0, 1\}$. Prove that the language $L = \{x \in \Sigma^* : \#_0(x) < \#_1(x)\}$ is irregular.

Solution:

Let D be an arbitrary DFA. Consider $S = \{0^n : n \geq 0\}$. Since S is infinite and D has finitely many states, we know $0^i \in S$ and $0^j \in S$ both end in the same state for some $i < j$. Append 1^j to both strings to get:

$a = 0^i 1^j$ Note that $a \in L$, because $i < j$ and $0^i 1^j \in \Sigma^*$.

$b = 0^j 1^j$ Note that $b \notin L$, because $j \not< j$.

Since a and b both end in the same state, and that state cannot both be an accept and reject state, D cannot recognize L. Since D was arbitrary, no DFA recognizes L; so, L is irregular.

2. Recurrences, Recurrences [15 points]

Define

$$T(n) = \begin{cases} n & \text{if } n = 0, 1 \\ 4T\left(\lfloor \frac{n}{2} \rfloor\right) + n & \text{otherwise} \end{cases}$$

Prove that $T(n) \leq n^3$ for $n \geq 3$.

Solution:

We go by strong induction on n. Let $P(n)$ be "$T(n) \leq n^3$" for $n \in \mathbb{N}$.

Base Cases. When $n = 3$, $T(3) = 4T\left(\lfloor \frac{3}{2} \rfloor\right) + 3 = 4T(1) + 3 = 7 \leq 27 = 3^3$.
When $n = 4$, $T(4) = 4T\left(\lfloor \frac{4}{2} \rfloor\right) + 4 = 4T(2) + 4 = 28 \leq 64 = 4^3$.
When $n = 5$, $T(5) = 4T\left(\lfloor \frac{5}{2} \rfloor\right) + 5 = 4T(2) + 5 = 29 \leq 4^4$.

Induction Hypothesis. Suppose $P(3) \wedge P(4) \wedge \cdots \wedge P(k)$ for some $k \geq 5$.

Induction Step. We want to prove $P(k+1)$: Note that

$$\begin{aligned}
T(k+1) &= 4T\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right) + k + 1, && \text{because } k+1 \geq 2. \\
&\leq 4\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right)^3 + k + 1, && \text{by IH.} \\
&\leq 4\left(\frac{k+1}{2}\right)^3 + k + 1, && \text{by def of floor.} \\
&= 4\left(\frac{(k+1)^3}{2^3}\right) + k + 1, && \text{by algebra.} \\
&= \frac{(k+1)^3}{2} + k + 1, && \text{by algebra.} \\
&= \frac{(k+1)((k+1)^2 + 2)}{2}, && \text{by algebra.} \\
&\leq \frac{(k+1)((k+1)^2 + (k+1)^2)}{2}, && \text{because } (k+1)^2 \geq 2. \\
&= (k+1)^3, && \text{by algebra}
\end{aligned}$$

Thus, since the base case and induction step hold, the $P(n)$ is true for $n \geq 3$.

3. All The Machines! [15 points]

Let $\Sigma = \{0, 1, 2\}$.
Consider $L = \{w \in \Sigma^* :$ Every 1 in the string has at least one 0 before and after it$\}$.

(a) Give a regular expression that represents A. **Solution:**

$(0 \cup 2)^*(0(0 \cup 1 \cup 2)^*0)^*(0 \cup 2)^*$

(b) Give a DFA that recognizes A. **Solution:**

Omitted.

(c) Give a CFG that generates A.

Solution:

$$
\begin{aligned}
S &\to 0S \mid 2S \mid T \\
T &\to 0R0T \mid X \\
R &\to 0 \mid 1 \mid 2 \\
X &\to 0X \mid 2X \mid \varepsilon
\end{aligned}
$$

4. Structural CFGs [15 points]

Consider the following CFG: $\mathbf{S} \to \varepsilon \mid \mathbf{SS} \mid \mathbf{S}1 \mid \mathbf{S}01$. Another way of writing the recursive definition of this set, Q, is as follows:

- $\varepsilon \in Q$
- If $S \in Q$, then $S1 \in Q$ and $S01 \in Q$
- If $S, T \in Q$, then $ST \in Q$.

Prove, by structural induction that if $w \in Q$, then w has at least as many 1's as 0's.

Solution:

We go by structal induction on w. Let $P(w)$ be "$\#_0(w) \leq \#_1(w)$" for $w \in \Sigma^*$.

Base Case. When $w = \varepsilon$, note that $\#_0(w) = 0 = \#_1(w)$. So, the claim is true.

Induction Hypothesis. Suppose $P(w), P(v)$ are true for some w, v generated by the grammar.

Induction Step 1. Note that $\#_0(w1) = \#_0(w) \leq \#_1(w) + 1 = \#_1(w1)$ by IH, and $\#_0(w01) = \#_0(w) + 1 \leq \#_1(w) + 1 = \#_1(w01)$ by IH.

Induction Step 2. Note that $\#_0(wv) = \#_0(w) + \#_0(v) \leq \#_1(w) + \#_1(v)$ by IH.

Since the claim is true for all recursive rules, the claim is true for all strings generated by the grammar.

5. Tralse! [15 points]

For each of the following answer True or False and give a short explanation of your answer.

(a) Any subset of a regular language is also regular. **Solution:**

False. Consider $\{0,1\}^*$ and $\{0^n1^n : n \geq 0\}$. Note that the first is regular and the second is irregular, but the second is a subset of the first.

(b) The set of programs that loop forever on at least one input is decidable. **Solution:**

False. If we could solve this problem, we could solve HaltNoInput. Intuitively, a program that solves this problem would have to try all inputs, but, since the program might infinte loop on some of them, it won't be able to.

(c) If $\mathbb{R} \subseteq A$ for some set A, then A is uncountable. **Solution:**

True. Diagonalization would still work; alternatively, if A were countable, then we could find an surjective function between $\mathbb{N}$ and $\mathbb{R}$ by skipping all the elements in A that aren't in $\mathbb{R}$.

(d) If the domain of discourse is people, the logical statement

$$\exists x \; (P(x) \to \forall y \; (x \neq y \to \neg P(y))$$

can be correctly translated as "There exists a unique person who has property P". **Solution:**

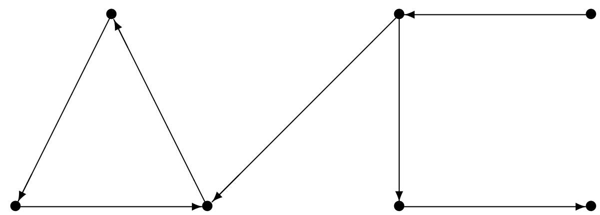
False. Any x for which $P(x)$ is false makes the entire statement true. This is not the same as there existing a unique person with property P.

(e) $\exists x \; (\forall y \; P(x,y)) \to \forall y \; (\exists x \; P(x,y))$ is true regardless of what predicate P is. **Solution:**

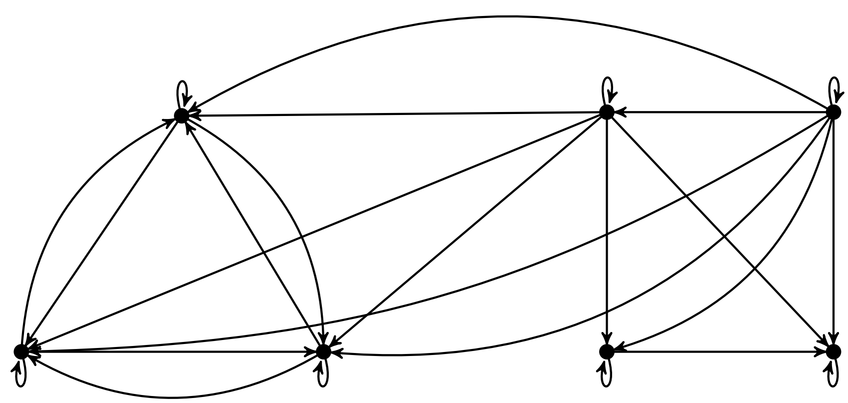
True. The left part of the implication is saying that there is a single x that works for all y; the right one is saying that for every y, we can find an x that depends on it, but the single x that works for everything will still work.

6. Relationships! [15 points]

The following is the graph of a binary relation R.



(a) Draw the transitive-reflexive closure of R. [5 points] **Solution:**



(b) Let $S = \{(X, Y) : X, Y \in \mathcal{P}(\mathbb{N}) \wedge X \subseteq Y\}$.

Recall that R is antisymmetric iff $((a, b) \in R \wedge a \neq b) \rightarrow (b, a) \notin R$.

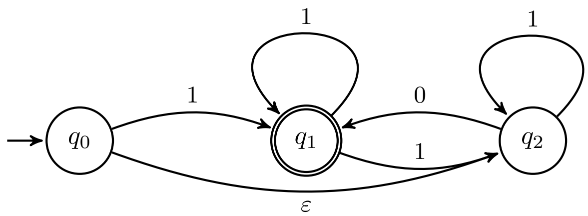
Prove that S is antisymmetric. [10 points]

Solution:

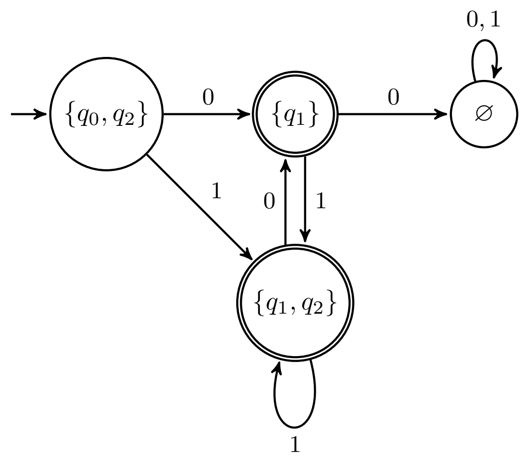
Suppose $(a, b) \in S$ and $a \neq b$. Then, by definition of S, $a \subset b$ and there is some $x \in b$ where $x \notin a$ (since they aren't equal). Then, $(b, a) \notin S$, because $b \not\subseteq a$, because $x \in b$ and $x \notin a$. So, S is antisymmetric.

7. Construction Paper! [15 points]

Convert the following NFA into a DFA using the algorithm from lecture.



Solution:



8. Modern DFAs [15 points]

Let $\Sigma = \{0, 1, 2\}$. Construct a DFA that recognizes exactly strings with a 0 in all positions i where $i \% 3 = 0$.

Solution: