CSE 311 : Midterm Exam Solutions

Instructions

- This is a take home exam. It is due 72 hours after it is released, i.e. at 11:59 PM on Sunday November 15th, Seattle time (GMT-8).
- Remember to follow the collaboration policy:
  - You are permitted to work with at most 3 other students in the class; you may discuss the exam only with those students and the course staff.
  - As always, you must write up your work independently.
- This exam is open-book/open-note. You may use any resource the staff has provided, as well as external resources.
- But you may not search for the particular problems in the exam.
- Submit your responses to gradescope. You may handwrite or typeset solutions (as long as they are legible). We do not recommend trying to write on this document; we have not left enough room.
- If you have a question, please check the pinned post on Ed where we've posted common clarifications. If you still have a question, please ask a **private** question on ed.
- Unless otherwise noted, we expect English proofs.

Advice

- Remember to properly format English proofs (e.g. introduce all of your variables).
- If you write a symbolic proof, follow the style guidelines on the webpage.
- We give some partial credit for the beginning and end of a proof. Even if you don't know how the middle goes, you can write the start of the proof and put the "target" and conclusion at the bottom.
- Take a deep breath. Gumball says you can do it.

Question	Max points
Translation Tricks	23
Warm-Ups	16
FUNctional Sets	25
Numbers	15
Induction	20
Survey	1
Total	**100**

1. Translation Tricks [23 points]

For the rest of this page, let your domain of discourse be "books and people". You may use the following predicates for the rest of this page:

- $\texttt{HasRead}(p, b)$ is true iff person p has read book b.
- $\texttt{Enjoyed}(p, b)$ is true iff person p enjoyed book b.
- $\texttt{Author}(p, b)$ is true iff person p is an author of book b.
- $\texttt{SciFi}(b)$ is true iff the book b is a science fiction book.
- $\texttt{KnowsDeMorgans}(x)$ is true iff x is a book where DeMorgan's Laws is stated or x is a person who knows DeMorgan's Laws.
- $\texttt{short}(x)$ is true iff x is a book that is less than 100 pages, or x is a person who is $5'6''$ or shorter.
- $\texttt{book}(b)$ is true iff b is a book.
- $\texttt{person}(p)$ is true iff p is a person.
- $=, \neq$ (with their usual meanings of "equal" and "not equal").

You may use the following two constants: Robbie, Hank Green

1.1. Reading is Fun! [16 points]

Translate each of these statements into symbolic logic (use predicates and quantifiers). In part (b), you should have one overall expression that represents the whole assertion.

(a) Robbie only enjoys science fiction books. **Solution:**

$\forall x\, \texttt{Enjoyed}(\texttt{Robbie}, x) \to \texttt{SciFi}(x)$
OR $\forall x(\texttt{Enjoyed}(\texttt{Robbie}, x) \wedge \texttt{book}(x) \to \texttt{SciFi}(x))$

(b) Both of the following are true:

- Robbie has read a book authored by Hank Green.
- Hank Green has read a book authored by Hank Green.

Solution:

$\exists x \exists y([\texttt{Author}(\texttt{Hank Green}, x) \wedge \texttt{HasRead}(\texttt{Robbie}, x)] \wedge [\texttt{Author}(\texttt{Hank Green}, y) \wedge \texttt{HasRead}(\texttt{Hank Green}, y)])$

(c) Hank Green is an author of exactly two books. **Solution:**

$\exists x \exists y(\texttt{Author}(\texttt{Hank Green}, x) \wedge \texttt{Author}(\texttt{Hank Green}, y) \wedge x \neq y \wedge [\forall z\, \texttt{Author}(\texttt{Hank Green}, z) \to (z = x \vee z = y)])$
OR $\exists x \exists y \forall z(\texttt{Author}(\texttt{Hank Green}, x) \wedge \texttt{Author}(\texttt{Hank Green}, y) \wedge x \neq y \wedge \wedge[\texttt{Author}(\texttt{Hank Green}, z) \to (z = x \vee z = y)])$
OR anything equivalent.

Note that $\exists x \exists y \forall z([\texttt{Author}(\texttt{Hank Green}, z)] \to [\texttt{Author}(\texttt{Hank Green}, x) \wedge \texttt{Author}(\texttt{Hank Green}, y) \wedge x \neq y])$ is NOT correct – if Hank Green has written no books, this evaluates to true.

(d) Every short book is enjoyed by someone who knows DeMorgan's Laws. **Solution:**

$\forall x \exists y([\texttt{short}(x) \wedge \texttt{book}(x)] \rightarrow [\texttt{enjoyed}(y,x) \wedge \texttt{person}(y) \wedge \texttt{KnowsDeMorgans}(y)])$

The initial upload for this solution was incorrect — the domain restriction on x is required.

1.2. Short Books and Short People [7 points]

(a) Translate $\forall x \exists y([\texttt{person}(x) \wedge \texttt{short}(x)] \rightarrow [\texttt{short}(y) \wedge \texttt{enjoyed}(x,y)])$ into English. Take advantage of domain restriction. [2 points] **Solution:**

Every short person enjoyed a short book.
OR for every short person, there is a short book that they enjoyed.

(b) Symbolically negate the sentence above. Simplify until negations apply only to individual predicates. Show your work (but you do not need to name the rules you are using). [3 points] **Solution:**

$\neg[\forall x \exists y([\texttt{person}(x) \wedge \texttt{short}(x)] \rightarrow [\texttt{short}(y)\, \texttt{enjoyed}(x,y)])]$
$\exists x \forall y \neg[([\texttt{person}(x) \wedge \texttt{short}(x)] \rightarrow [\texttt{short}(y) \wedge \texttt{enjoyed}(x,y)])]$
$\exists x \forall y(\texttt{person}(x) \wedge \texttt{short}(x) \wedge [\neg\, \texttt{short}(y) \vee \neg\, \texttt{enjoyed}(x,y)])$

(c) Write the English version of the negation of the original sentence. Your English translation should take advantage of domain restriction (even if your symbols do not). [2 points]

Solution:

There is a short person who does not enjoy any short books.
OR There is a short person such that for all short books, they do not enjoy the book.

Note that "There is a short person who doesn't enjoy all short books" is usually interpreted in English as $\exists x \exists y(\texttt{person}(x) \wedge \texttt{person}(y) \wedge \texttt{short}(x) \wedge \texttt{short}(y) \wedge \neg\, \texttt{Enjoys}(x,y))$ which is not the same.

2. Warm-Ups [16 points]

Recall that a number x is rational if there are integers p and q such that $q \neq 0$ and $x = p/q$.
Equivalently, in symbols: $\texttt{rational}(x) = \exists p \exists q(x = p/q \wedge \texttt{Integer}(p) \wedge \texttt{Integer}(q) \wedge q \neq 0)$.
Call x "irrational" if it is not rational.
For the rest of this page, let your domain of discourse be real numbers. You may use the following predicates $\texttt{rational}(x)$ (as defined above), $\texttt{irrational}(x)$ to mean x is irrational, and $\texttt{Integer}(x)$ to mean "x is an integer."

Consider the statement "If the product of two numbers is irrational, then at least one of the two numbers is irrational."

(a) Write this statement with quantifiers and predicate notation. [3 points] **Solution:**

$\forall x \forall y(\texttt{irrational}(xy) \rightarrow \texttt{irrational}(x) \vee \texttt{irrational}(y)$

(b) Prove the statement. You may give an English proof **or** a (symbolic) inference proof.
Give only one proof not both. If you give the symbolic proof, you may use "algebra" as a justification and any rules on the inference rules and logical connectives reference sheets. [9 points]

Solution:

We argue by contrapositive.
Suppose x and y are arbitrary rational numbers. Then by definition $x = p/q$ and $y = s/t$ where $p, q, s, t \in \mathbb{Z}$, $q \neq 0$, and $t \neq 0$.

Consider $xy = \frac{p}{q} \cdot \frac{s}{t} = \frac{ps}{qt}$. Observe that ps and qt are integers since integers are closed under multiplication. Moreover, $qt \neq 0$ since $q \neq 0$ and $t \neq 0$. By the definition of rational, xy is rational, as required.

Solution:

Assume, for the sake of contradiction, that there are numbers x, y such that xy is irrational but x and y are both rational. Then by definition $x = p/q$ and $y = s/t$ where $p, q, s, t \in \mathbb{Z}$, $q \neq 0$, and $t \neq 0$.

Consider $xy = \frac{p}{q} \cdot \frac{s}{t} = \frac{ps}{qt}$. Observe that ps and qt are integers since integers are closed under multiplication. Moreover, $qt \neq 0$ since $q \neq 0$ and $t \neq 0$. By the definition of rational, xy is rational.

But we supposed xy is rational. That's a contradiction! So we have thatif the product of two numbers is irrational, then at least one of the two numbers is irrational.

Solution:

1.Let x be arbitrary	–
2.Let y be arbitrary	–
3.1 $\texttt{Rational}(x) \wedge \texttt{Rational}(y)$	Assumption
3.2 $\texttt{Rational}(x)$	Elim $\wedge$ (3.1)
3.3 $\texttt{Rational}(y)$	Elim $\wedge$ (3.1)
3.4 $\exists a \exists b(x = a/b \wedge \texttt{Integer}(a) \wedge \texttt{Integer}(b) \wedge b \neq 0)$	Definition of $\texttt{Rational}$ (3.1)
3.5 $\exists b(x = p/b \wedge \texttt{Integer}(p) \wedge \texttt{Integer}(b) \wedge b \neq 0)$	Elim $\exists$ (3.4)
3.6 $x = p/q \wedge \texttt{Integer}(p) \wedge \texttt{Integer}(q) \wedge q \neq 0)$	Elim $\exists$ (3.5)
3.7 $\exists a \exists b(y = a/b \wedge \texttt{Integer}(a) \wedge \texttt{Integer}(b) \wedge b \neq 0)$	Definition of $\texttt{Rational}$ (3.1)
3.8 $\exists b(y = s/b \wedge \texttt{Integer}(s) \wedge \texttt{Integer}(b) \wedge b \neq 0)$	Elim $\exists$ (3.7)
3.9 $y = s/t \wedge \texttt{Integer}(s) \wedge \texttt{Integer}(t) \wedge t \neq 0)$	Elim $\exists$ (3.8)
3.10 $x = p/q$	Elim $\wedge$ (3.5)
3.11 $y = s/t$	Elim $\wedge$ (3.9)
3.12 $xy = (ps)/(qt)$	Algebra (3.10-3.11)
3.13 $Integer(pq)$	The product of integers is integers, 3.5, 3.8
3.13 $Integer(st)$	The product of integers is integers, 3.5, 3.8
3.14 $st \neq 0$	The product on nonzero numbers is nonzero 3.5, 3.8
3.15 $xy = (ps)/(qt) \wedge \texttt{Integer}(pq) \wedge \texttt{Integer}(st) \wedge st \neq 0$	Intro $\wedge$ (multiple times, 3.13-3.15)
3.16 $\texttt{Rational}(xy)$	
4. $\texttt{Rational}(x) \wedge \texttt{Rational}(y) \rightarrow \texttt{Rational}(xy)$	Definition of rational
5. $\neg\texttt{Rational}(xy) \rightarrow \neg(\texttt{Rational}(x) \wedge \texttt{Rational}(y))$	Law of Contrapositive
6. $\neg\texttt{Rational}(xy) \rightarrow \neg\texttt{Rational}(x) \vee \neg\texttt{Rational}(y)$	DeMorgan's Law
7. $\texttt{Irrational}(xy) \rightarrow \texttt{Irrational}(x) \vee \texttt{Irrational}(y)$	definition of irrational
8. $\forall w\, \texttt{Irrational}(xw) \rightarrow \texttt{Irrational}(x) \vee \texttt{Irrational}(w)$	Intro $\forall$
9. $\forall z \forall w\, \texttt{Irrational}(zw) \rightarrow \texttt{Irrational}(z) \vee \texttt{Irrational}(w)$	Intro $\forall$

(c) Give the converse of the original statement in symbols and English.[1] [4 points]

Solution:

If at least one of two numbers is irrational, then their product is irrational.

$\forall x \forall y(\texttt{irrational}(x) \vee \texttt{irrational}(y) \rightarrow \texttt{irrational}(xy))$.

[1]That is, take the converse of the implication in the statement, leaving the quantifiers unchanged

3. FUNctional Sets [25 points]

For any function f and any set S, define $\text{output}(f, S) = \{y : \exists x(x \in S \wedge f(x) = y)\}$

For example, $\text{output}(x^2, \{1, 2, 3\}) = \{y : \exists x(x \in \{1, 2, 3\} \wedge f(x) = y)\} = \{1, 4, 9\}$, and $\text{output}(x^2, \{-1, 1\}) = \{1\}$.

(a) Show that $\text{output}(f, A \cap B) \subseteq \text{output}(f, A) \cap \text{output}(f, B)$. [10 points]

Solution:

Let A, B be arbitrary sets and f be an arbitrary function. Let y be an arbitrary element of $\text{output}(f, A \cap B)$. By definition of output, there is an x in $A \cap B$ such that $f(x) = y$. Note that $x \in A$ and $x \in B$. Since $x \in A$, $f(x) = y \in \text{output}(f, A)$ by definition of output and $f(x) = y \in \text{output}(f, B)$ since $x \in B$. Thus $y \in [\text{output}(f, A) \cap \text{output}(f, B)]$ by definition of intersection.

(b) These two sets are not equal in general. Convince yourself of that fact by considering $A = \{x : x \in \mathbb{Z} \wedge x < 3\}$, $B = \{x \in \mathbb{Z} \wedge x > -3\}$ and $f(x) = x^2$. You do not have to write anything for this part. [0 points] **Solution:**

$\text{output}(f, A \cap B) = \text{output}\left(f, \{-2, -1, 0, 1, 2\}\right) = \{0, 1, 4\}$ but $\text{output}(f, A) = \mathbb{N}$ and $\text{output}(f, B) = \mathbb{N}$ so $\text{output}(f, A) \cap \text{output}(f, B) = \mathbb{N}$.

(c) Call a function "non-repetitive" if $\forall x \forall y(f(x) = f(y) \rightarrow x = y)$. Show that x^2 is **not** "non-repetitive" if your domain of discourse is all integers. [3 points]

Solution:

Consider $x = -3, y = 3$. $f(-3) = (-3^2) = 9$ and $f(3) = 3^2 = 9 = f(-3)$, so the hypothesis is met, but $-3 \neq 3$.

(d) Show that if a f is non-repetitive, then $\text{output}(f, A) \cap \text{output}(f, B) \subseteq \text{output}(f, A \cap B)$. [12 points]

Solution:

Suppose f is an arbitrary non-repetitive function. Let y be an arbitrary element of $\text{output}(f, A) \cap \text{output}(f, B)$. By definition of intersection, $y \in \text{output}(f, A)$ and $y \in \text{output}(f, B)$. By definition of output there is an $x \in A$ such that $f(x) = y$ and an $x' \in B$ such that $f(x') = y$. Observe that $f(x) = f(x')$, so by definition of non-repetitive $x = x'$. Thus $x \in A$ and $x \in B$ so $x \in A \cap B$, and by definition of output, $f(x) = y \in \text{output}(f, A \cap B)$

4. We Went a Little Overboard with the twos... [15 points]

Let a, b be arbitrary integers, and n be an arbitrary positive integer. The following statements look a lot like mod identities, but only one is actually true. Read them carefully.

For the false statement (part a), give an English proof showing it is false.
For the true statement (part b), give an English proof showing it is true.

(a) if $a \equiv b \pmod{n}$ then $a^2 \equiv b^2 \pmod{n^2}$. [3 points] **Solution:**

The claim is false. Consider $a = 1, b = 4, n = 3$. $1 \equiv 4 \pmod{3}$, but $1 \not\equiv 16 \pmod{9}$.

(b) If $a \equiv b \pmod{n}$ then $2a \equiv 2b \pmod{2n}$. [12 points] **Solution:**

Let a, b be arbitrary integers, n be an arbitrary positive integer. Suppose $a \equiv b \pmod{n}$. By definition of mod, $n|(b-a)$ and $nk = b - a$ for some integer k by the definition of divides. Muliplying both sides by 2, we have $2nk = 2b - 2a$. So $2n|(2b - 2a)$ by the definition of divides, and $2a \equiv 2b \pmod{n}$.

5. It's a Logical Deduction to Use Induction [20 points]

You have n pairs of people to put into a line. In each pair, one person wears a purple hat and the other wears a gold hat. As you put the people in a line, you need to ensure all of the following hold.

- The members of every pair are adjacent.
- The person at the front of the line wears a purple hat.
- The person at the back of the line wears a purple hat.

Call a line "properly ordered" if it meets all of those requirements.
Show that for all $n \geq 2$, in every properly ordered line there are two people wearing gold hats next to each other.

You must do this proof by induction. You may not use proof by contradiction (either for the whole proof or to argue the base case or inductive step).

Solution:

Let $P(n)$ be "A line of n pairs that is properly ordered has two consecutive people in gold hats." We show $P(n)$ for all $n \geq 2$ by induction on n.

Base Case: $n = 2$ The only line meeting these conditions is (P,G), (G,P); the two people in gold hats are neighbors.

Inductive Hypothesis: Let k be an arbitrary integer at least 2. Suppose $P(k)$ (i.e. suppose any properly ordered line with k pairs has consecutive gold-hat-wearing people).

Inductive Step: Consider an arbitrary properly ordered line with $k + 1$ pairs. Look at the second pair in the list. We have two cases:

Case 1: They are in (G,P) order. The first two pairs are (P,G) (G,P), so we have two adjacent gold-hat-wearing-people.

Case 2: They are in (P,G) order. Then the second pair to the end: starts and ends with a purple hat and therefore meet all the conditions for a good seating order. By Inductive hypthesis, there are adjacent gold-hat-wearing people in the remaining list (and thus in the whole list).

By the principle of induction, we have $P(n)$ for all $n \geq 2$. That is every properly ordered line has at least two people wearing gold hats next to each other.

6. Help us help future you [1 points]

This is the first time 311 has given takehome exams, for our sake (when writing the final) and for the sake of future instructors, it would help to know these pieces of information:

(a) How many hours did you work on this total (estimate to the nearest half-hour)? **Solution:**

> We estimated the stats, if you're curious:
> About 25% took 5 hours or less
> About 50% took 7 hours or less
> About 75% took 9 hours or less

(b) How much of that time was just making a readable submission (typing or cleanly writing – estimate to the nearest half-hour)?

(c) How many people did you work with?

(d) On homeworks this quarter, how many people have you usually worked with?

(e) How did you feel about the exam?

6.1. Drawing

Remember T'Vin? The Vulcan child you adopted on HW1? How do you think he's doing? Tell us or draw a picture (any nonempty piece of paper will get credit for this question; looking at these help the TAs stay happy while grading :))