CSE 311: Foundations of Computing I

Section 8: CFGs, Relations, and DFAs Solutions

1. CFGs

Construct CFGs for the following languages:

(a) All binary strings that end in 00.

Solution:

$$\mathbf{S} \rightarrow 0\mathbf{S} \mid 1\mathbf{S} \mid 00$$

(b) All binary strings that contain at least three 1's.

Solution:

$$\mathbf{S} \rightarrow \mathbf{TTT}$$
$$\mathbf{T} \rightarrow 0\mathbf{T} \mid \mathbf{T}0 \mid 1\mathbf{T} \mid 1$$

(c) All binary strings with an equal number of 1's and 0's.

Solution:

$$\mathbf{S} \rightarrow 0\mathbf{S}1\mathbf{S} \mid 1\mathbf{S}0\mathbf{S} \mid \varepsilon$$

and

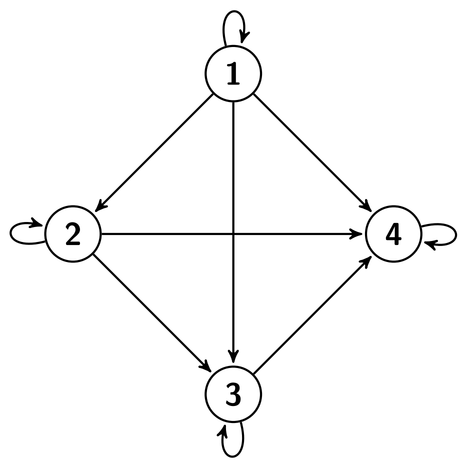
$$\mathbf{S} \rightarrow \mathbf{SS} \mid 0\mathbf{S}1 \mid 1\mathbf{S}0 \mid \varepsilon$$

both work. Note: The fact that all the strings generated have the property is easy to show (by induction) but the fact that one can generate all strings with the property is trickier. To argue this that each of these is grammars is enough one would need to consider how the difference between the # of 0's seen and the # of 1's seen occurs in prefixes of any string with the property.

2. Relations

(a) Draw the transitive-reflexive closure of $\{(1,2),(2,3),(3,4)\}$.

Solution:



(b) Suppose that R is reflexive. Prove that $R \subseteq R^2$.

Solution:

Suppose $(a,b) \in R$. Since R is reflexive, we know $(b,b) \in R$ as well. Since there is a b such that $(a,b) \in R$ and $(b,b) \in R$, it follows that $(a,b) \in R^2$. Thus, $R \subseteq R^2$.

(c) Consider the relation $R = \{(x,y) \;:\; x = y+1\}$ on $\mathbb{N}$. Is R reflexive? Transitive? Symmetric? Anti-symmetric?

Solution:

It isn't reflexive, because $1 \neq 1+1$; so, $(1,1) \notin R$. It isn't symmetric, because $(2,1) \in R$ (because $2 = 1+1$), but $(1,2) \notin R$, because $1 \neq 2+1$. It isn't transitive, because note that $(3,2) \in R$ and $(2,1) \in R$, but $(3,1) \notin R$. It is anti-symmetric, because consider $(x,y) \in R$ such that $x \neq y$. Then, $x = y+1$ by definition of R. However, $(y,x) \notin R$, because $y = x-1 \neq x+1$.

(d) Consider the relation $S = \{(x,y) \;:\; x^2 = y^2\}$ on $\mathbb{R}$. Prove that S is reflexive, transitive, and symmetric.

Solution:

Consider $x \in \mathbb{R}$. Note that by definition of equality, $x^2 = x^2$; so, $(x,x) \in R$; so, R is reflexive.

Consider $(x,y) \in R$. Then, $x^2 = y^2$. It follows that $y^2 = x^2$; so, $(y,x) \in R$. So, R is symmetric.

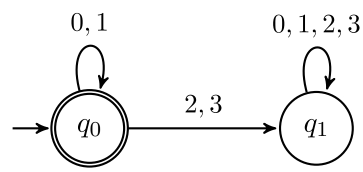
Suppose $(x,y) \in R$ and $(y,z) \in R$. Then, $x^2 = y^2$, and $y^2 = z^2$. Since equality is transitive, $x^2 = z^2$. So, $(x,z) \in R$. So, R is transitive.

3. DFAs

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1, 2, 3\}$.

(a) All binary strings.

Solution:

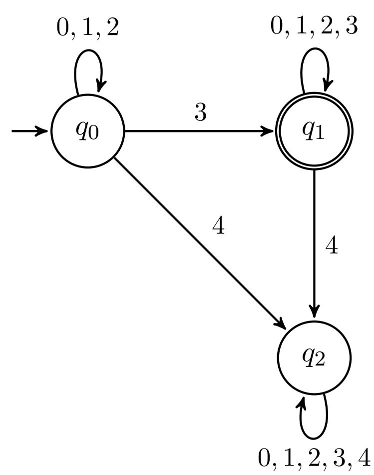


q_0**:** binary strings

q_1**:** strings that contain a character which is not 0 or 1.

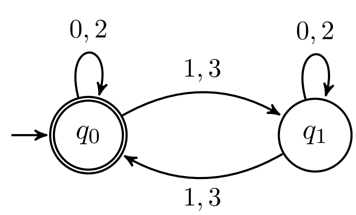
(b) All strings that contain at least one 3 but no 4.

Solution:



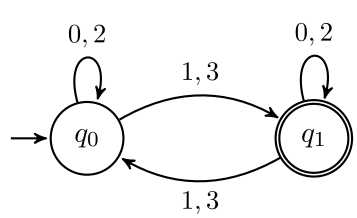
(c) All strings whose digits sum to an even number.

Solution:



(d) All strings whose digits sum to an odd number.

Solution:



4. Powers of Relations

Let A be a set and R a relation on A. Use induction to prove that R^n is exactly the pairs of elements from A that are connected by a path of length n in the graph $G = (A, R)$.

Solution:

Let $P(n)$ be "for all $a, b \in A$, $(a, b) \in R^n$ iff there is a path of length n from a to b in the graph G". We will prove $P(n)$ for all integers $n \in \mathbb{N}$ by induction.

Base Case $(n = 0)$**:** $R^0 = \{(a, a) \mid a \in A\}$ and every element is and can only be connected to itself by a path of length 0, so $P(0)$ holds.

Induction Hypothesis: Assume that $P(k)$ holds for an arbitrary $k \in \mathbb{N}$.

Induction Step: Our goal is to show $P(k + 1)$. I.e., that R^{n+1} is exactly the set of pairs connected by a path of length $n + 1$ in G. Note that, by definition, $R^{n+1} = \{(a, c) \mid \exists b \in A\,((a, b) \in R^n \wedge (b, c) \in R)\}$.

By the induction hypothesis, $(a, b) \in R^n$ holds iff a and b are connected by a path of length n in G. Hence, if $(a, c) \in R^{n+1}$, then there is some $b \in A$ such that there is a path $a = v_0, v_1, \ldots, v_n = b$ in G, which means that $a = v_0, v_1, \ldots, b, c$ is a path of length $n + 1$ connecting a and c in G.

On the other hand, suppose that $a = v_0, v_1, \ldots, v_n, v_{n+1} = c$ is a path of length $n + 1$ connecting a and c in G. By the definition of a path, this means that $(v_n, c) \in R$. Furthermore, since $a = v_0, v_1, \ldots, v_n$ is a path of length n in G, by the inductive hypothesis, we have $(a, v_n) \in R^n$. Thus, by the definition of R^{n+1}, we can see that $(a, c) \in R^{n+1}$.

Conclusion: $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.