CSE 311: Foundations of Computing I

Section 3: Predicate Logic and Inference Solutions

1. Quantifier Switch

Consider the following pairs of sentences. For each pair, determine if one implies the other (or if they are equivalent).

(a) $\forall x\, \forall y\, P(x, y)$ $\qquad$ $\forall y\, \forall x\, P(x, y)$

Solution:

These sentences are the same; switching universal quantifiers makes no difference.

(b) $\exists x\, \exists y\, P(x, y)$ $\qquad$ $\exists y\, \exists x\, P(x, y)$

Solution:

These sentences are the same; switching existential quantifiers makes no difference.

(c) $\forall x\, \exists y\, P(x, y)$ $\qquad$ $\forall y\, \exists x\, P(x, y)$

Solution:

These are only the same if P is symmetric (e.g. the order of the arguments doesn't matter). If the order of the arguments does matter, then these are different statements. For instance, if $P(x, y)$ is "$x < y$, then the first statement says "for every x, there is a corresponding y such that $x < y$, whereas the second says "for every y, there is a corresponding x such that $x < y$. In other words, in the first statement y is a function of x, and in the second x is a function of y.

(d) $\forall x\, \exists y\, P(x, y)$ $\qquad$ $\exists x\, \forall y\, P(x, y)$

Solution:

These two statements are usually different.

(e) $\forall x\, \exists y\, P(x, y)$ $\qquad$ $\exists y\, \forall x\, P(x, y)$

Solution:

The second statement implies the former. Here is a proof:

1.	$\exists y\, \forall x\, P(x, y)$	[Given]
2.	$\forall x\, P(x, a)$	[Elim $\exists$: 1 (some a)]
3.	Let z be arbitrary	
4.	$P(z, a)$	[Elim $\forall$: 2]
5.	$\exists y\, P(z, y)$	[Intro $\exists$: 4]
10.	$\forall x\, \exists y\, P(x, y)$	[Intro $\forall$: 3, 5]

2. Formal Proof I

Show that $\neg p \to s$ follows from $p \vee q$, $q \to r$ and $r \to s$.

Solution:

1.	$p \vee q$		[Given]
2.	$q \to r$		[Given]
3.	$r \to s$		[Given]
	4.1. $\neg p$	[Assumption]	
	4.2. q	[Elim of $\vee$: 1, 4.1]	
	4.3. r	[MP of 4.2, 2]	
	4.4. s	[MP 4.3, 3]	
4.	$\neg p \to s$		[Direct Proof Rule]

3. Formal Spoofs

For each of of the following proofs, determine why the proof is incorrect. Then, consider whether the conclusion of the proof is true or not. If it is true, state how the proof could be fixed. If it is false, give a counterexample.

(a) Show that $p \to (q \vee r)$ follows from $p \to q$ and r.

1.	$p \to q$	[Given]
2.	r	[Given]
3.	$p \to (q \vee r)$	[$\vee$ Intro: 1, 2]

(b) Show that q follows from $\neg p \vee q$ and p.

1.	$\neg p \vee q$	[Given]
2.	p	[Given]
3.	q	[$\vee$ Elim: 1, 2]

(c) Show that q follows from $q \vee p$ and $\neg p$.

1.	$\neg p$	[Given]
2.	$q \vee p$	[Given]
3.	$q \vee \mathsf{F}$	[Substitute $p = \mathsf{F}$ since $\neg p$ holds: 1, 2]
4.	q	[Identity: 3]

(d) Show that $\exists z \, \forall x \, P(x, z)$ follows from $\forall x \, \exists y \, P(x, y)$.

1.	$\forall x \, \exists y \, P(x, y)$	[Given]
2.	$\forall x \, P(x, c)$	[$\exists$ Elim: 1, c special]
3.	$\exists z \, \forall x \, P(x, z)$	[$\exists$ Intro: 2]

(e) Show that $\exists z \, (P(z) \wedge Q(z))$ follows from $\forall x \, P(x)$ and $\exists y \, Q(y)$.

1.	$\forall x \, P(x)$	[Given]
2.	$\exists y \, Q(y)$	[Given]
3.	Let z be arbitrary	
4.	$P(z)$	[$\forall$ Elim: 1]
5.	$Q(z)$	[$\exists$ Elim: 2, let z be the object that satisfies $Q(z)$]
6.	$P(z) \wedge Q(z)$	[$\wedge$ Intro: 4, 5]
7.	$\exists z \, (P(z) \wedge Q(z))$	[$\exists$ Intro: 6]

Solution:

The mistakes are as follows:

(a) Line 3: inference rule used on a subexpression.

(b) Line 3: $\vee$ Elim requires $\neg\neg p$ not p.

(c) Line 3: there is no such "substitute for" rule

(d) Line 2: inference rule used on a subexpression.

(e) Line 5: must use a new variable with $\exists$ Elim

Next, we consider whether the statements are actually true:

(a) **True**. Since r is true, $q \vee r$ is true. Hence, the latter is true if we assume p. (It's true even if we don't assume p.) This can be formalized using the direct proof rule.

(b) **True**. Add a line inferring $\neg\neg p$ from p.

(c) **True**. Line 4 follows instead by $\vee$ Elim.

(d) **False**. Let the domain of discourse be the integers and define $P(x, y)$ to be $x < y$. Then, the hypothesis is true: for every integer, there is a larger integer. However, the conclusion is false: there is integer that is larger than every other integer (no largest integer). Hence, there can be no correct proof that the conclusion follows from the hypothesis.

(e) **True**. Remove line 3 declaring z to be arbitrary. Instead, define z by applying $\exists$ Elim to line 2. Finally, move line 4 after line 5. (We need z to already exist in order to apply $\forall$ Elim, so we need to do this after the $\exists$ Elim on line for, which actually defines z.)

4. Formal Proof II

Show that $\neg p$ follows from $\neg(\neg r \vee t)$, $\neg q \vee \neg s$ and $(p \to q) \wedge (r \to s)$.

Solution:

1.	$\neg(\neg r \vee t)$	[Given]
2.	$\neg q \vee \neg s$	[Given]
3.	$(p \to q) \wedge (r \to s)$	[Given]
4.	$\neg\neg r \wedge \neg t$	[DeMorgan's Law: 1]
5.	$\neg\neg r$	[Elim of $\wedge$: 4]
6.	r	[Double Negation: 5]
7.	$r \to s$	[Elim of $\wedge$: 3]
8.	s	[MP: 6,7]
9.	$\neg\neg s$	[Double Negation: 8]
10.	$\neg q$	[Elim of $\vee$: 2, 9]
11.	$p \to q$	[Elim of $\wedge$: 3]
12.	$\neg q \to \neg p$	[Contrapositive: 11]
13.	$\neg p$	[MP: 10, 12]