CSE 311: Foundations of Computing I

Section 2: Equivalences and Predicate Logic Solutions

1. Equivalences

Prove that each of the following pairs of propositional formulae are equivalent using propositional equivalences.

(a) $p \leftrightarrow q$ $\qquad$ $(p \land q) \lor (\neg p \land \neg q)$

Solution:

$$
\begin{aligned}
p \leftrightarrow q &\equiv (p \to q) \land (q \to p) && \text{[iff is two implications]} \\
&\equiv (\neg p \lor q) \land (q \to p) && \text{[Law of Implication]} \\
&\equiv (\neg p \lor q) \land (\neg q \lor p) && \text{[Law of Implication]} \\
&\equiv ((\neg p \lor q) \land \neg q) \lor ((\neg p \lor q) \land p) && \text{[Distributivity]} \\
&\equiv (\neg q \land (\neg p \lor q)) \lor ((\neg p \lor q) \land p) && \text{[Commutativity]} \\
&\equiv ((\neg q \land \neg p) \lor (\neg q \land q)) \lor ((\neg p \lor q) \land p) && \text{[Distributivity]} \\
&\equiv ((\neg p \land \neg q) \lor (\neg q \land q)) \lor ((\neg p \lor q) \land p) && \text{[Commutativity]} \\
&\equiv ((\neg p \land \neg q) \lor (q \land \neg q)) \lor ((\neg p \lor q) \land p) && \text{[Commutativity]} \\
&\equiv ((\neg p \land \neg q) \lor (q \land \neg q)) \lor (p \land (\neg p \lor q)) && \text{[Commutativity]} \\
&\equiv ((\neg p \land \neg q) \lor (q \land \neg q)) \lor ((p \land \neg p) \lor (p \land q)) && \text{[Distributivity]} \\
&\equiv ((\neg p \land \neg q) \lor F) \lor ((p \land \neg p) \lor (p \land q)) && \text{[Negation]} \\
&\equiv ((\neg p \land \neg q) \lor F) \lor (F \lor (p \land q)) && \text{[Negation]} \\
&\equiv (\neg p \land \neg q) \lor (F \lor (p \land q)) && \text{[Identity]} \\
&\equiv (\neg p \land \neg q) \lor ((p \land q) \lor F) && \text{[Commutativity]} \\
&\equiv (\neg p \land \neg q) \lor (p \land q) && \text{[Identity]} \\
&\equiv (p \land q) \lor (\neg p \land \neg q) && \text{[Commutativity]}
\end{aligned}
$$

(b) $\neg p \to (q \to r)$ $\qquad$ $q \to (p \lor r)$

Solution:

$$
\begin{aligned}
\neg p \to (q \to r) &\equiv \neg\neg p \lor (q \to r) && \text{[Law of Implication]} \\
&\equiv p \lor (q \to r) && \text{[Double Negation]} \\
&\equiv p \lor (\neg q \lor r) && \text{[Law of Implication]} \\
&\equiv (p \lor \neg q) \lor r && \text{[Associativity]} \\
&\equiv (\neg q \lor p) \lor r && \text{[Commutativity]} \\
&\equiv \neg q \lor (p \lor r) && \text{[Associativity]} \\
&\equiv q \to (p \lor r) && \text{[Law of Implication]}
\end{aligned}
$$

2. Non-equivalence

Prove that the following pairs of propositional formulae are not equivalent by finding inputs they differ on.

(a) $p \to q$ $\qquad\qquad$ $q \to p$

Solution:

A: They differ when $p = \mathsf{T}$ and $q = \mathsf{F}$.

B: They differ when $p = \mathsf{T}$ and $q = \mathsf{F}$ since $p \to q = \mathsf{T} \to \mathsf{F} \equiv \mathsf{F}$ but $q \to p \equiv \mathsf{F} \to \mathsf{T} \equiv \mathsf{T}$.

(b) $p \to (q \wedge r)$ $\qquad\qquad$ $(p \to q) \wedge r$

Solution:

A: They differ when $p = r = \mathsf{F}$ since $p \to (q \wedge r) = \mathsf{F} \to (q \wedge \mathsf{F}) \equiv \mathsf{T}$ but $(p \to q) \wedge r = (\mathsf{F} \to q) \wedge \mathsf{F} \equiv \mathsf{F}$.

B: They differ when $p = r = \mathsf{F}$ since $p \to (q \wedge r) = \mathsf{F} \to (q \wedge \mathsf{F}) \equiv \mathsf{F} \to \mathsf{F} \equiv \mathsf{T}$ and on the other hand $(p \to q) \wedge r = (\mathsf{F} \to q) \wedge \mathsf{F} \equiv \mathsf{T} \wedge \mathsf{F} \equiv \mathsf{F}$.

3. Boolean Algebra

For each of the following parts, write the logical expression using boolean algebra operators. Then, simplify it using axioms and theorems of boolean algebra.

(a) $\neg p \vee (\neg q \vee (p \wedge q))$

Solution:

A: First, we replace $\neg, \vee$, and $\wedge$. This gives us $p' + q' + pq$. (Note that the parentheses are not necessary in boolean algebra since the operations are associative.) Next, we can use DeMorgan's laws to get the slightly simpler $(pq)' + pq$. Then, we can use commutativity to get $pq + (pq)'$ and complementarity to get 1. (Note that this is another way of saying the formula is a tautology.)

B: Replacing $\neg$, $\vee$, and $\wedge$ gives us $p' + q' + pq$. (Note that parentheses are not necessary in boolean algebra since the operations are associative.) Then, we can simplify as follows:

$$\begin{aligned} p' + q' + pq &= (pq)' + pq && \text{De Morgan} \\ &= pq + (pq)' && \text{Commutativity} \\ &= 1 && \text{Complementarity} \end{aligned}$$

Since 1 in boolean algebra means T, this shows that the original expression is a tautology.

(b) $\neg(p \vee (q \wedge p))$

Solution:

A:

$$\begin{aligned} (p + qp)' &= p'(qp)' && \text{De Morgan} \\ &= p'(q' + p') && \text{De Morgan (on second term)} \\ &= p'(p' + q') && \text{Commutativity} \\ &= p' && \text{Absorption} \end{aligned}$$

B: Translating to Boolean algebra, we get $(p + qp)'$. Then, we can simplify as follows:

$$\begin{aligned} (p + qp)' &= p'(qp)' && \text{De Morgan} \\ &= p'(q' + p') && \text{De Morgan} \\ &= p'(p' + q') && \text{Commutativity} \\ &= p' && \text{Absorption} \end{aligned}$$

4. Canonical Forms

Consider the boolean functions $F(A, B, C)$ and $G(A, B, C)$ specified by the following truth table:

A	B	C	$F(A, B, C)$	$G(A, B, C)$
1	1	1	1	0
1	1	0	1	1
1	0	1	0	0
1	0	0	0	0
0	1	1	1	1
0	1	0	1	0
0	0	1	0	1
0	0	0	1	0

(a) Write the DNF and CNF expressions for $F(A, B, C)$.

Solution:

DNF: $ABC + ABC' + A'BC + A'BC' + A'B'C'$
CNF: $(A' + B + C')(A' + B + C)(A + B + C')$

(b) Write the DNF and CNF expressions for $G(A, B, C)$.

Solution:

DNF: $ABC' + A'BC + A'B'C$
CNF: $(A' + B' + C')(A' + B + C')(A' + B + C)(A + B' + C)(A + B + C)$