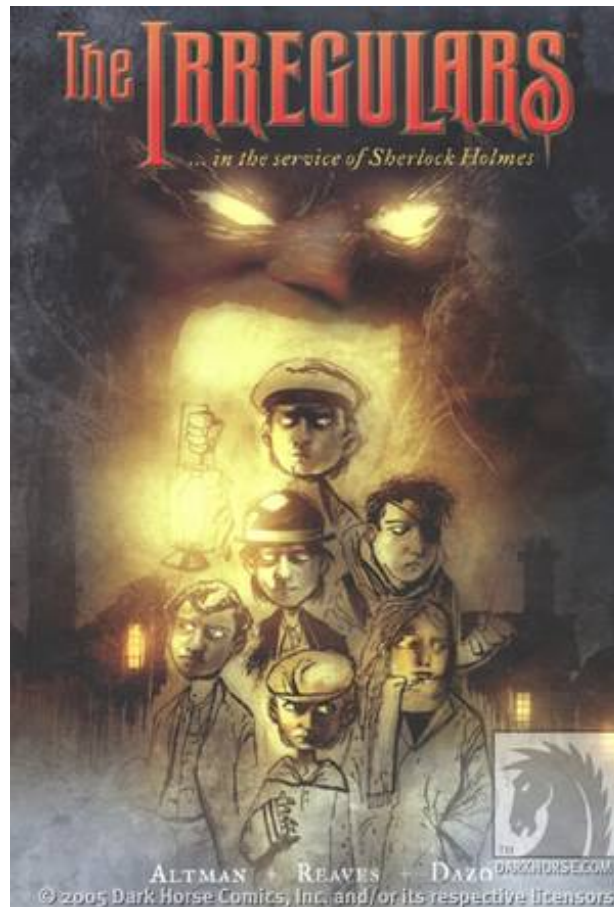
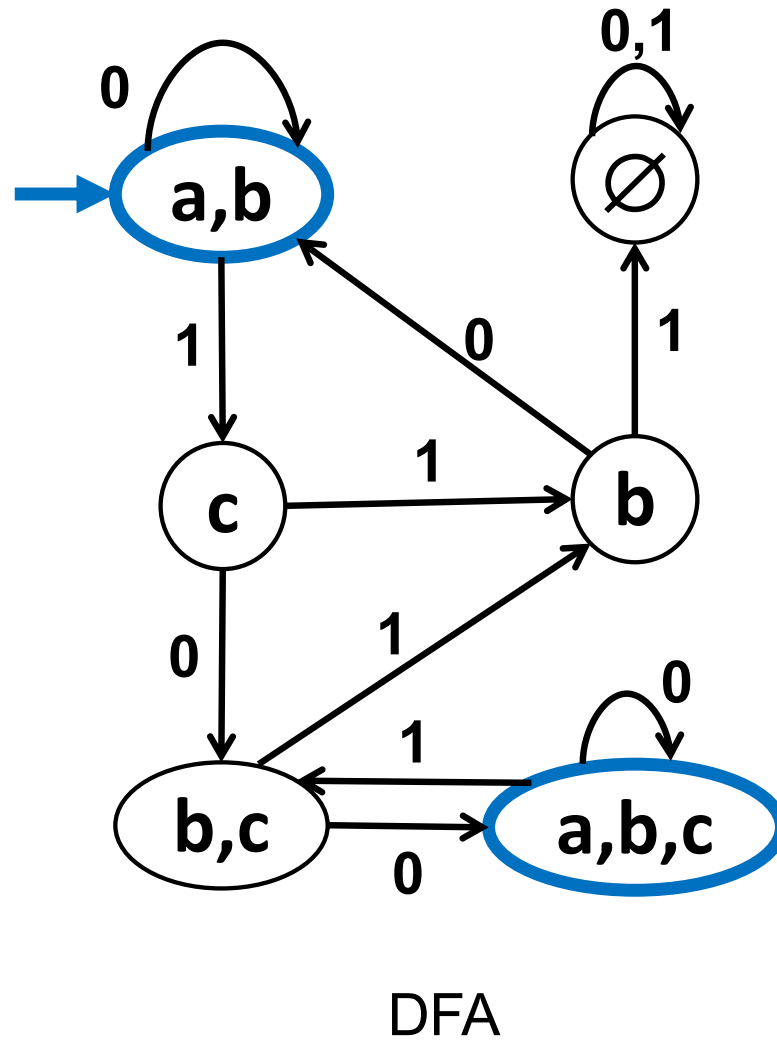
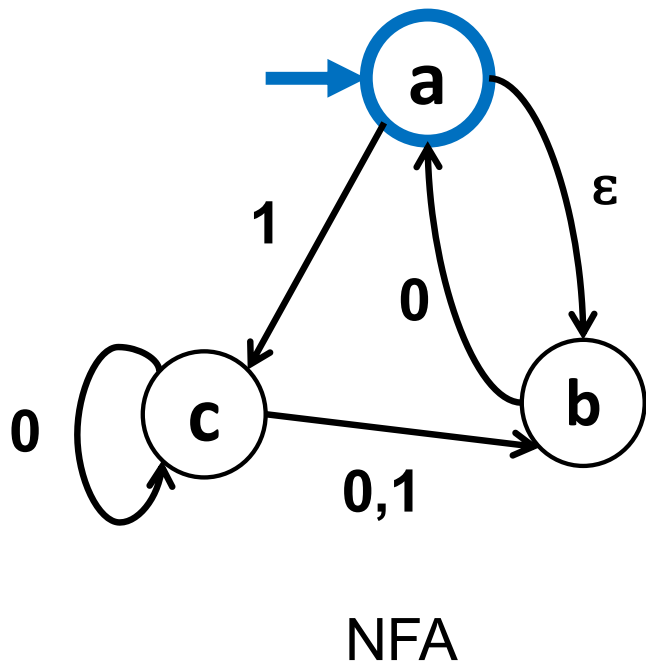


CSE 311: Foundations of Computing

Lecture 25: Languages vs Representations: Limitations of Finite Automata and Regular Expressions



Last time: NFA to DFA

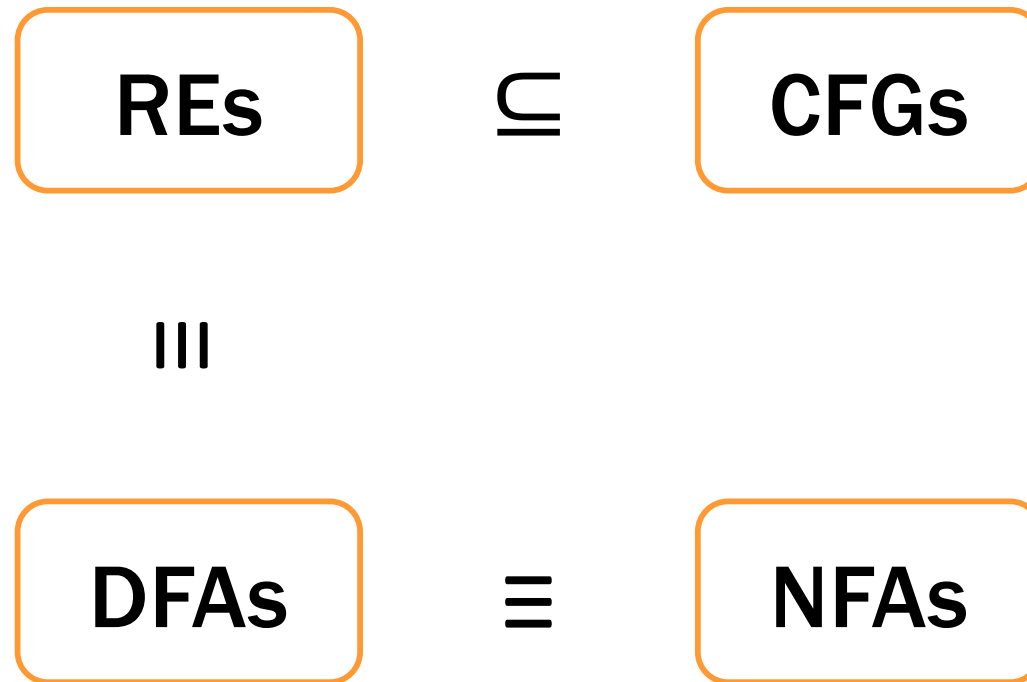


Exponential Blow-up in Simulating Nondeterminism

- In general the DFA might need a state for every subset of states of the NFA
 - Power set of the set of states of the NFA
 - n -state NFA yields DFA with at most 2^n states
 - We saw an example where roughly 2^n is necessary
 - “Is the n^{th} char from the end a 1?”

The famous “P=NP?” question asks whether a similar blow-up is always necessary to get rid of nondeterminism for polynomial-time algorithms

The story so far...



Last time: DFAs \equiv NFAs \equiv Regular expressions

We have shown how to build an optimal DFA for every regular expression

- Build NFA
- Convert NFA to DFA using subset construction
- Minimize resulting DFA

Theorem: A language is recognized by a DFA (or NFA) if and only if it has a regular expression

You need to know this fact but you don't need to know and we won't ask you anything about the construction for the "only if" direction from DFA/NFA to regular expression.

Languages represented by DFA, NFAs, or regular expressions are called **Regular Languages**

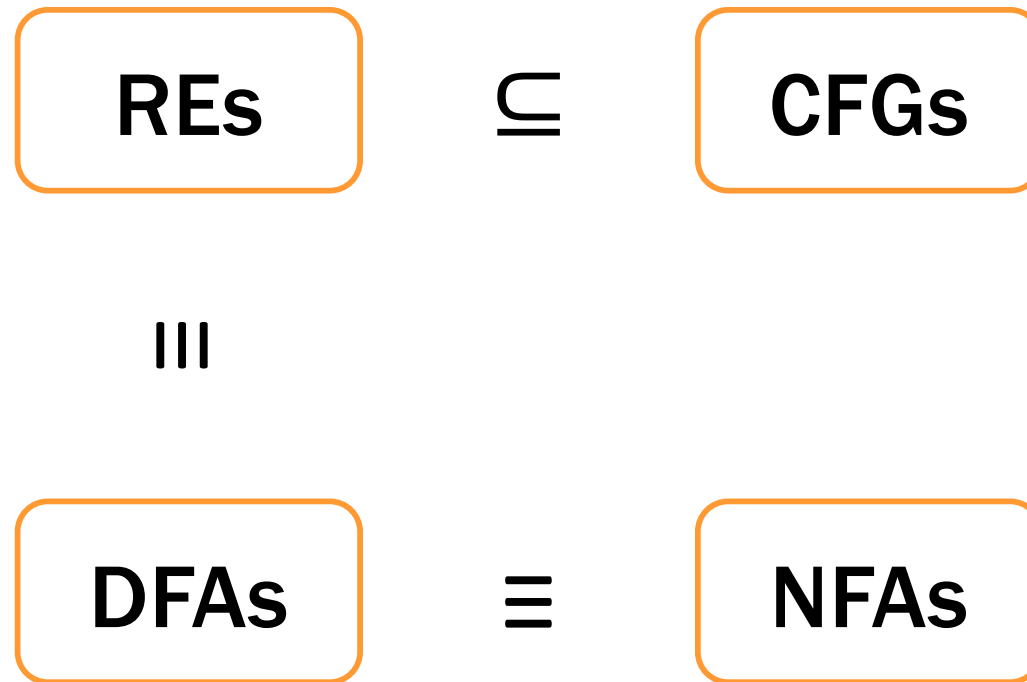
Application of FSMs: Pattern matching

- **Given**
 - a string s of n characters
 - a pattern p of m characters
 - usually $m \ll n$
- **Find**
 - all occurrences of the pattern p in the string s
- **Obvious algorithm:**
 - try to see if p matches at each of the positions in s
stop at a failed match and try matching at the next
position: $O(mn)$ running time.

Application of FSMs: Pattern Matching

- With DFAs can do this in $O(m + n)$ time.
- See Extra Credit problem on HW8 for some ideas of how to get to $O(m^2 + n)$.
- Even more general idea in practice: implemented in regular expression pattern matchers like grep:
 - Convert regular expression pattern to an NFA
 - Start building the equivalent DFA from the NFA using the subset construction but do this “on the fly”: only add arcs that are actually followed by the input text

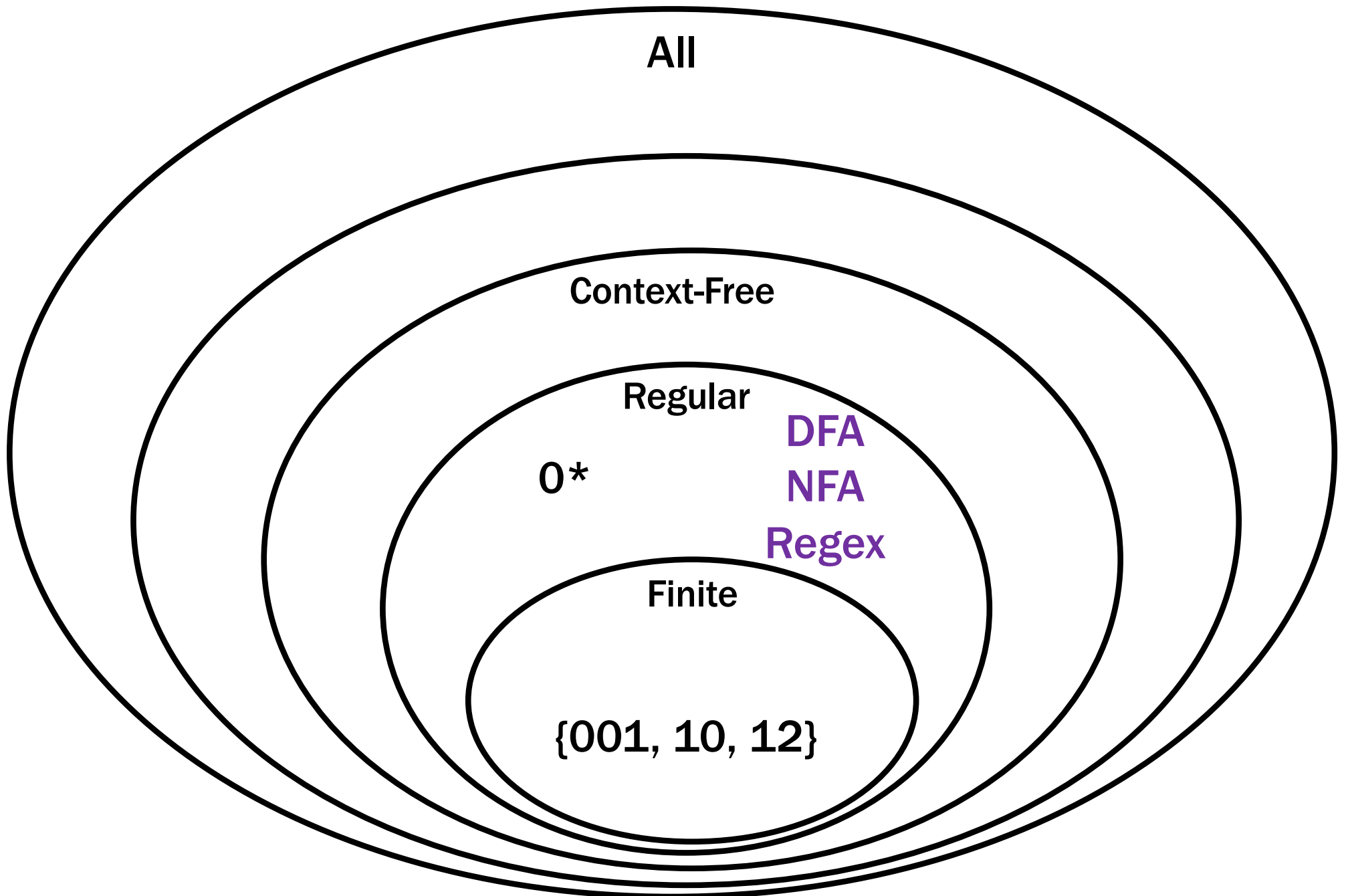
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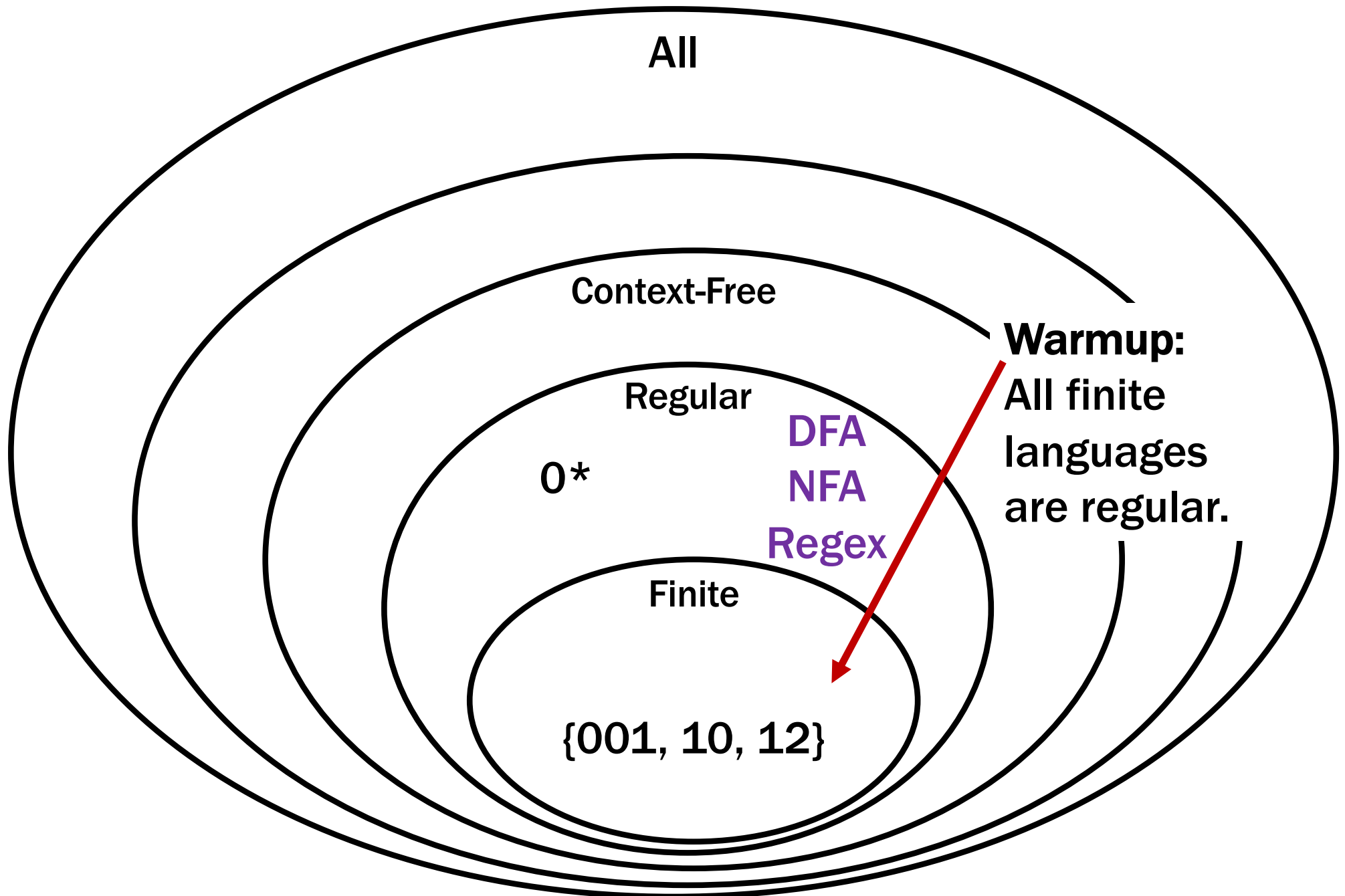
What languages have DFAs? CFGs?

All of them?

Languages and Representations!



Languages and Representations!



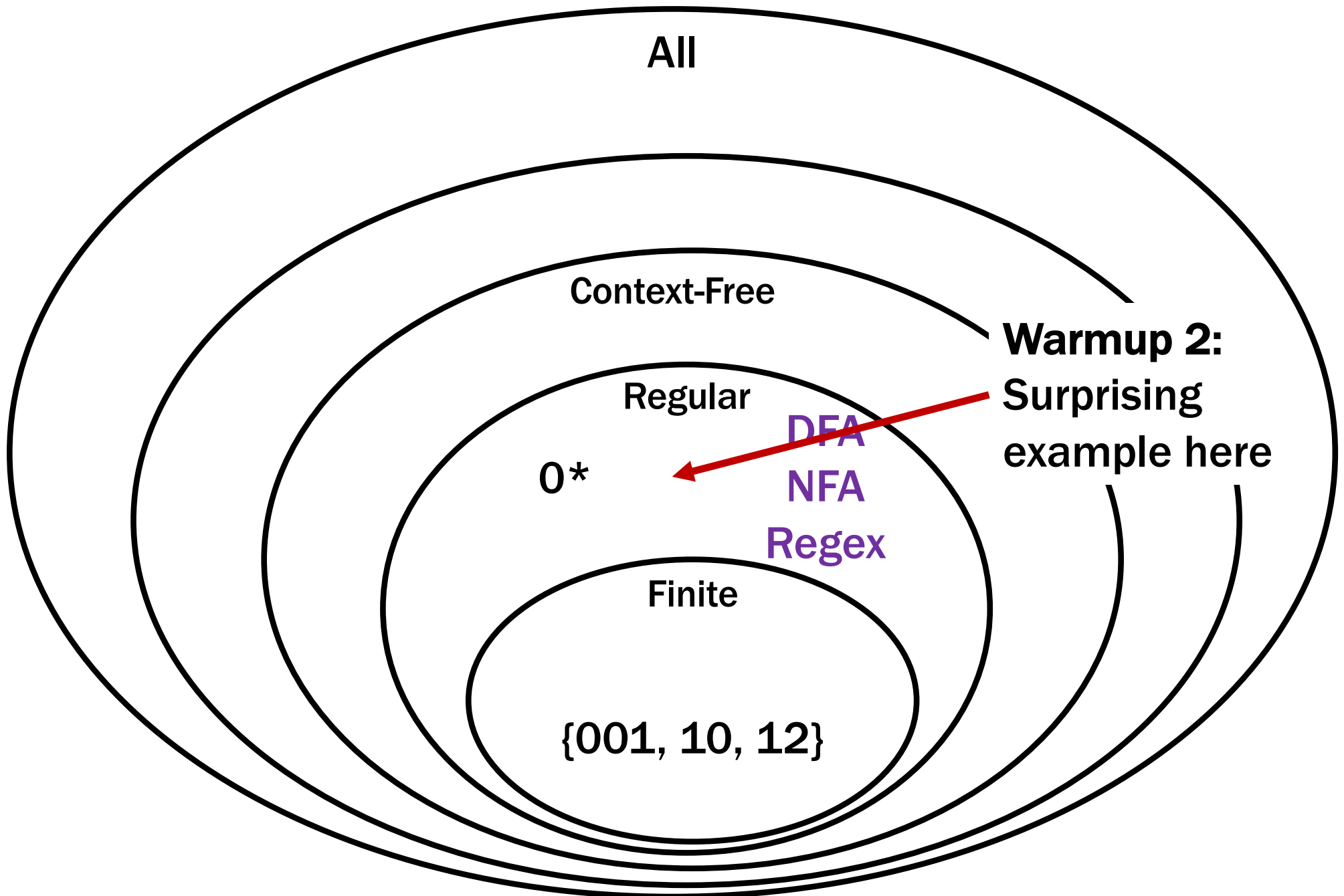
DFAs Recognize Any Finite Language

DFAs Recognize Any Finite Language

Construct a DFA for each string in the language.

Then, put them together using the union construction.

Languages and Machines!



An Interesting Infinite Regular Language

$L = \{x \in \{0, 1\}^* : x \text{ has an equal number of substrings } 01 \text{ and } 10\}$.

L is infinite.

0, 00, 000, ...

L is regular. How could this be?

That seems to require comparing counts...

- easy for a CFG (see section: strings with equal # of 0s and 1s)
- but seems hard for DFAs!

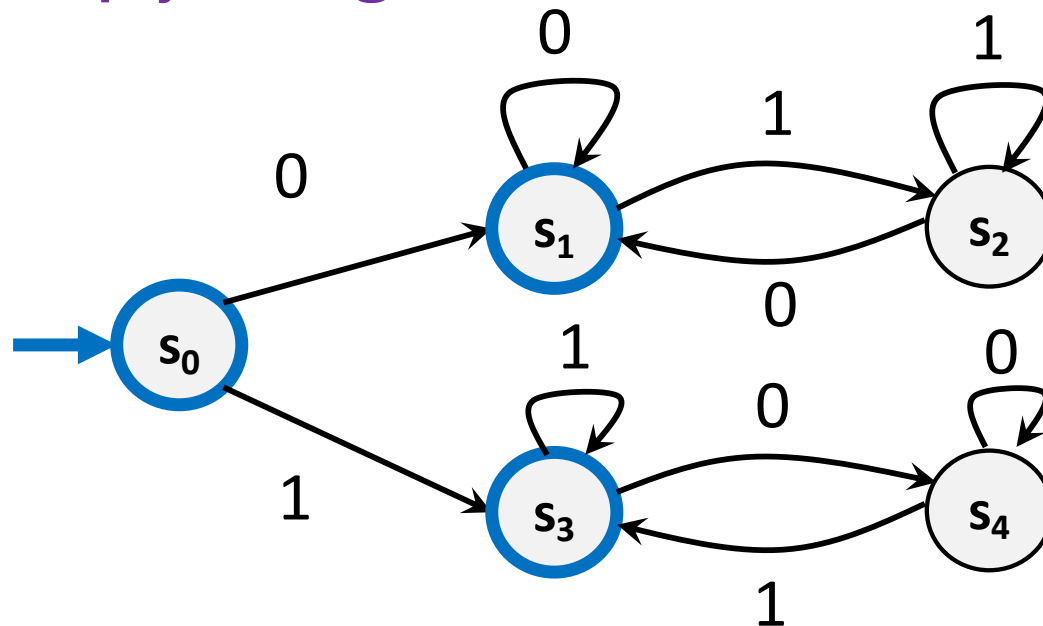
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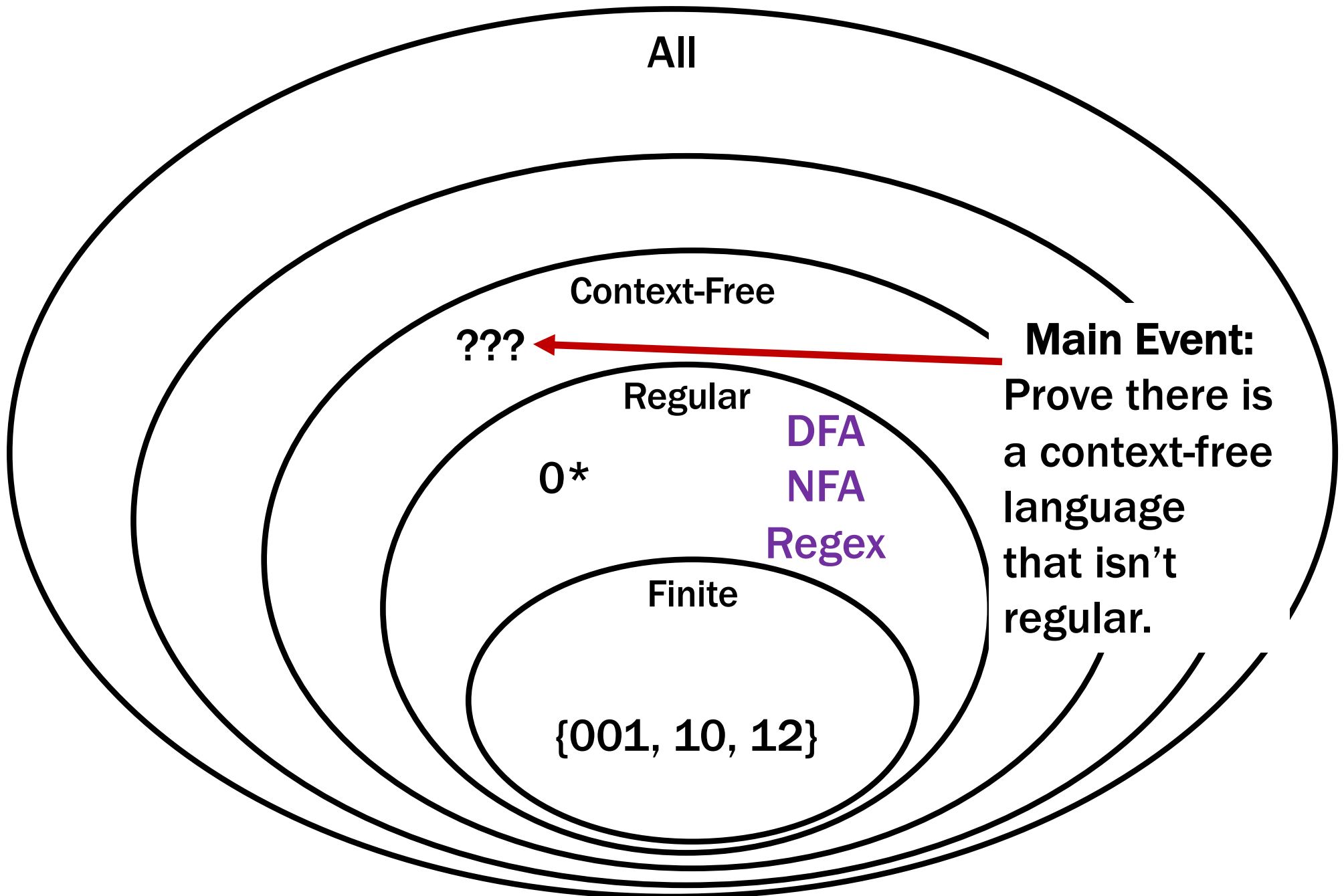
L is infinite.

0, 00, 000, ...

L is regular. How could this be? It is just the set of binary strings that are empty or begin and end with the same character!



Languages and Representations!



The language of “Binary Palindromes” is Context-Free

$$S \rightarrow \varepsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$$

Is the language of “Binary Palindromes” Regular ?

Intuition (NOT A PROOF!):

Q: What would a DFA need to keep track of to decide?

A: It would need to keep track of the “first part” of the input in order to check the second part against it

...but there are an infinite # of possible first parts and we only have finitely many states.

Proof idea: any machine that does not remember the entire first half will be wrong for some inputs

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that it's possible.
- Therefore, some DFA (call it **M**) exists that recognizes **B**

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- Our goal is to show that **M** must be “confused”... we want to show that it “does the wrong thing”.

How can a DFA be “wrong”?

- when it accepts or rejects a string it shouldn't.

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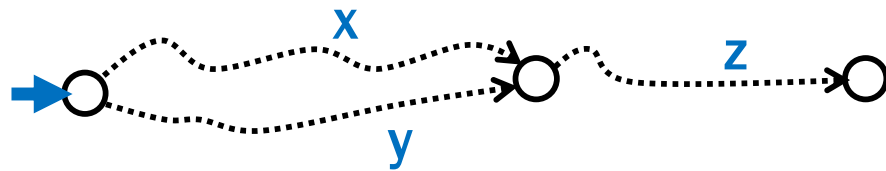
“**M** recognizes **B**” AND “**M** doesn't recognize **B**”,
which is a contradiction

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Key Idea 1: If two strings “collide” at any point, a DFA can no longer distinguish between them!

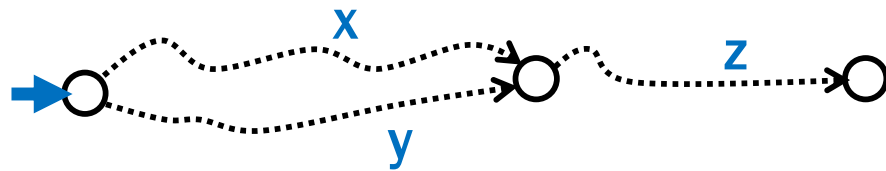


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Key Idea 2: Our machine **M** has a finite number of states which means if we have infinitely many strings, two of them must collide!

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We choose an **INFINITE** set **S** of “partial strings”
(which we intend to complete later).

1_____

01_____

001_____

0001_____

00001_____

.....

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- We want to show: **M** accepts or rejects a string it shouldn't.

We choose an **INFINITE** set **S** of “partial strings” (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an “accept” completion that the two strings **DO NOT SHARE**.

1_____

01_____

001_____

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.....

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, **M**, recognizes **B**.

We show **M** accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \dots\} = \{0^n1 : n \geq 0\}$.

Key Idea 2: Our machine has a finite number of states which means if we have infinitely many strings, two of them must collide!

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We show **M** accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \dots\} = \{0^n1 : n \geq 0\}$.

*Since there are finitely many states in **M** and infinitely many strings in **S**, there exist strings $0^a1 \in S$ and $0^b1 \in S$ with $a \neq b$ that end in the same state of **M**.*

SUPER IMPORTANT POINT: You do not get to choose what **a** and **b** are. Remember, we've just proven they exist...we have to take the ones we're given!

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, **M**, accepts **B**.

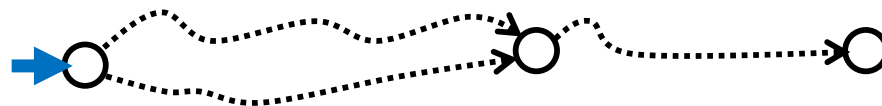
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Now, consider appending 0^a to both strings.

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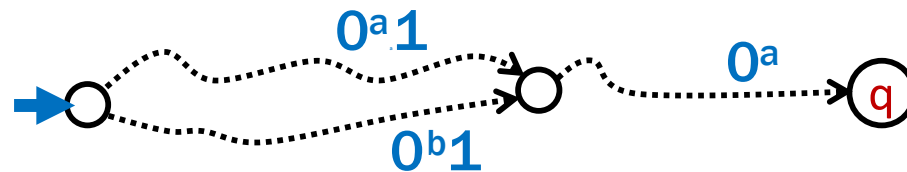
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Then, since 0^a1 and 0^b1 end in the same state, 0^a10^a and 0^b10^a also end in the same state, call it q .

But then **M** makes a mistake: q needs to be an accept state since $0^a10^a \in B$, but **M** would accept $0^b10^a \notin B$ which is an error.

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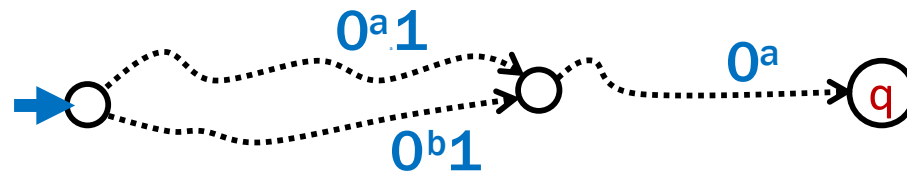
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*This is a contradiction since we assumed that **M** recognizes **B**. Since **M** was arbitrary, no DFA recognizes **B**.*

Showing that a Language L is not regular

1. “Suppose for contradiction that some DFA M recognizes L .”
2. Consider an INFINITE set S of “partial strings” (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an “accept” completion that the two strings DO NOT SHARE.
3. “Since S is infinite and M has finitely many states, there must be two strings s_a and s_b in S for $s_a \neq s_b$ that end up at the same state of M .”
4. Consider appending the (correct) completion t to each of the two strings.
5. “Since s_a and s_b both end up at the same state of M , and we appended the same string t , both $s_a t$ and $s_b t$ end at the same state q of M . Since $s_a t \in L$ and $s_b t \notin L$, M does not recognize L .”
6. “Since M was arbitrary, no DFA recognizes L .”

Prove $A = \{0^n 1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M , recognizes A .

Let $S =$

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Consider appending 1^a to both strings.

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Consider appending 1^a to both strings.

Note that $0^a 1^a \in A$, but $0^b 1^a \notin A$ since $a \neq b$. But they both end up in the same state of M , call it q . Since $0^a 1^a \in A$, state q must be an accept state but then M would incorrectly accept $0^b 1^a \notin A$ so M does not recognize A .

Since M was arbitrary, no DFA recognizes A .

Prove $P = \{\text{balanced parentheses}\}$ is not regular

Suppose for contradiction that some DFA, M , accepts P .

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Prove $P = \{\text{balanced parentheses}\}$ is not regular

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Let $S = \{(^n : n \geq 0)\}$. Since S is infinite and M has finitely many states, there must be two strings, $(^a$ and $(^b$ for some $a \neq b$ that end in the same state in M .

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Since M was arbitrary, no DFA recognizes P .

Showing that a Language L is not regular

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Fact: This method is optimal

- Suppose that for a language L , the set S is a *largest* set of “partial strings” with the property that for every pair $s_a \neq s_b \in S$, there is some string t such that one of $s_a t, s_b t$ is in L but the other isn't.
- If S is infinite, then L is not regular
- If S is finite, then the minimal DFA for L has precisely $|S|$ states, one reached by each member of S .

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- If S is infinite, then L is not regular
- If S is finite, then the minimal DFA for L has precisely $|S|$ states, one reached by each member of S .

Corollary: Our minimization algorithm was correct.

- we separated exactly those states for which some t would make one accept and another not accept

Fact: This method is optimal

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BTW: There is another method commonly used to prove languages not regular called the Pumping Lemma that we won't use in this course. Note that it doesn't always work.