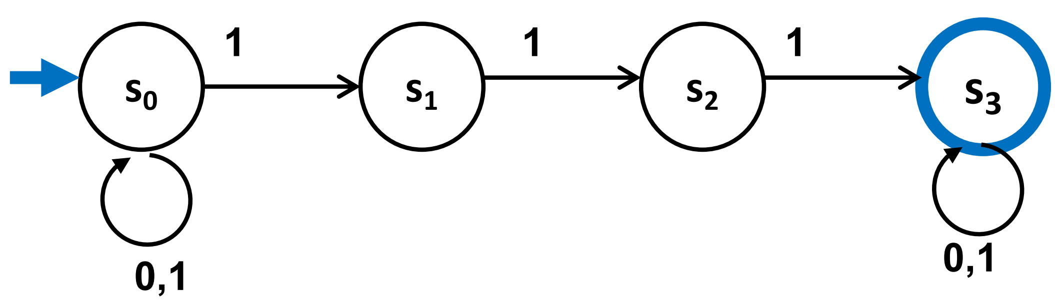
CSE 311: Foundations of Computing

Lecture 24: NFAs, Regular expressions, and NFA→DFA

Last time: Nondeterministic Finite Automata (NFA)

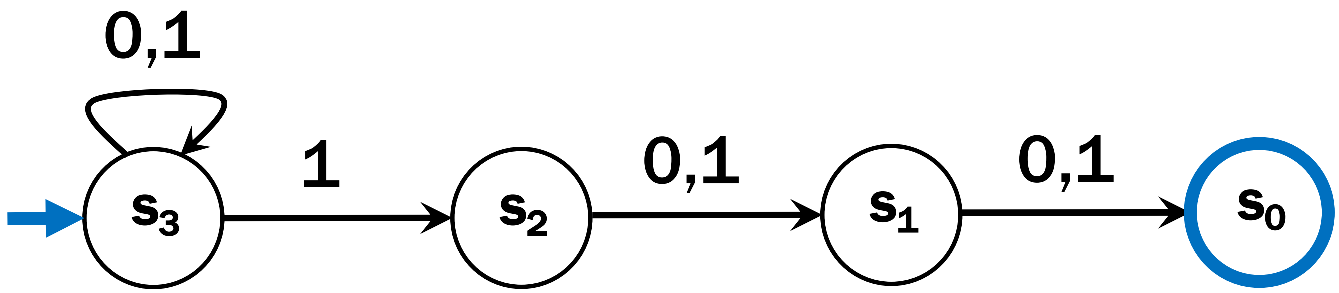
- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol— can have 0 or >1
  - Also can have edges labeled by empty string ε
- **Def:** x is in the language recognized by an NFA if and only if x labels a path from the start state to some final state



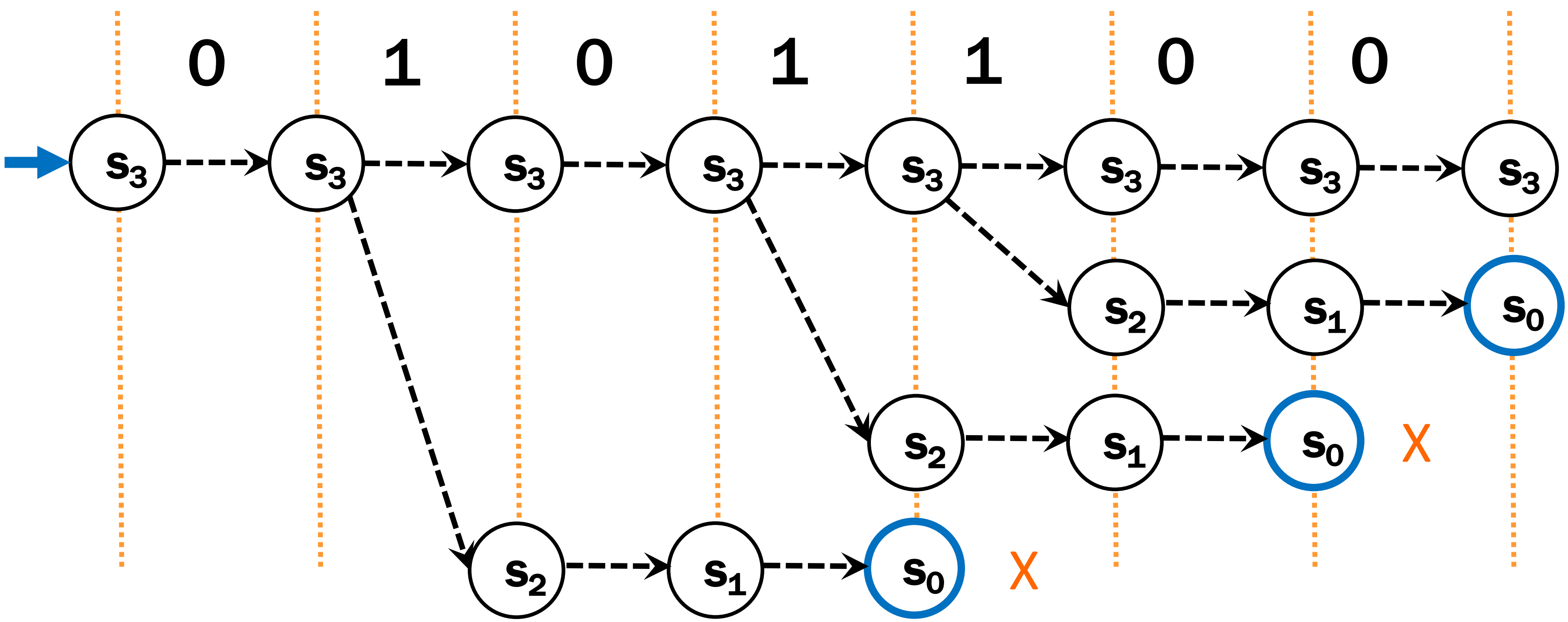
Last time: Three ways of thinking about NFAs

- Outside observer: Is there a path labeled by x from the start state to some final state?
- Perfect guesser: The NFA has input x and whenever there is a choice of what to do it magically guesses a good one (if one exists)
- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

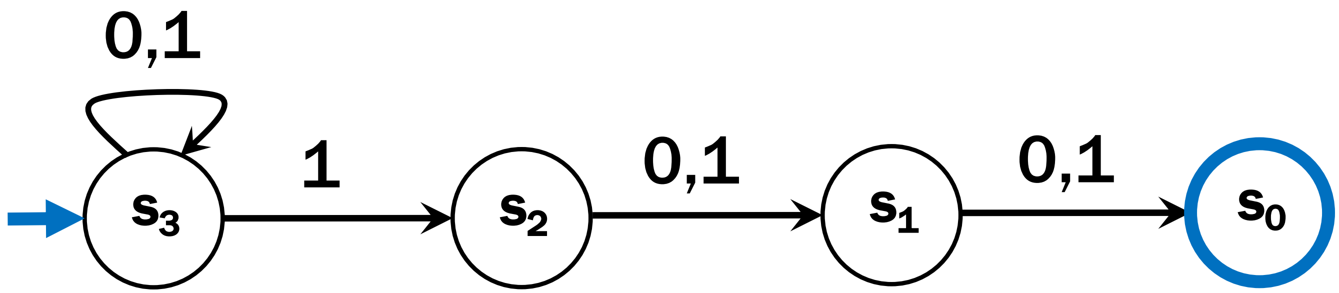
Last time: Parallel Exploration view of an NFA

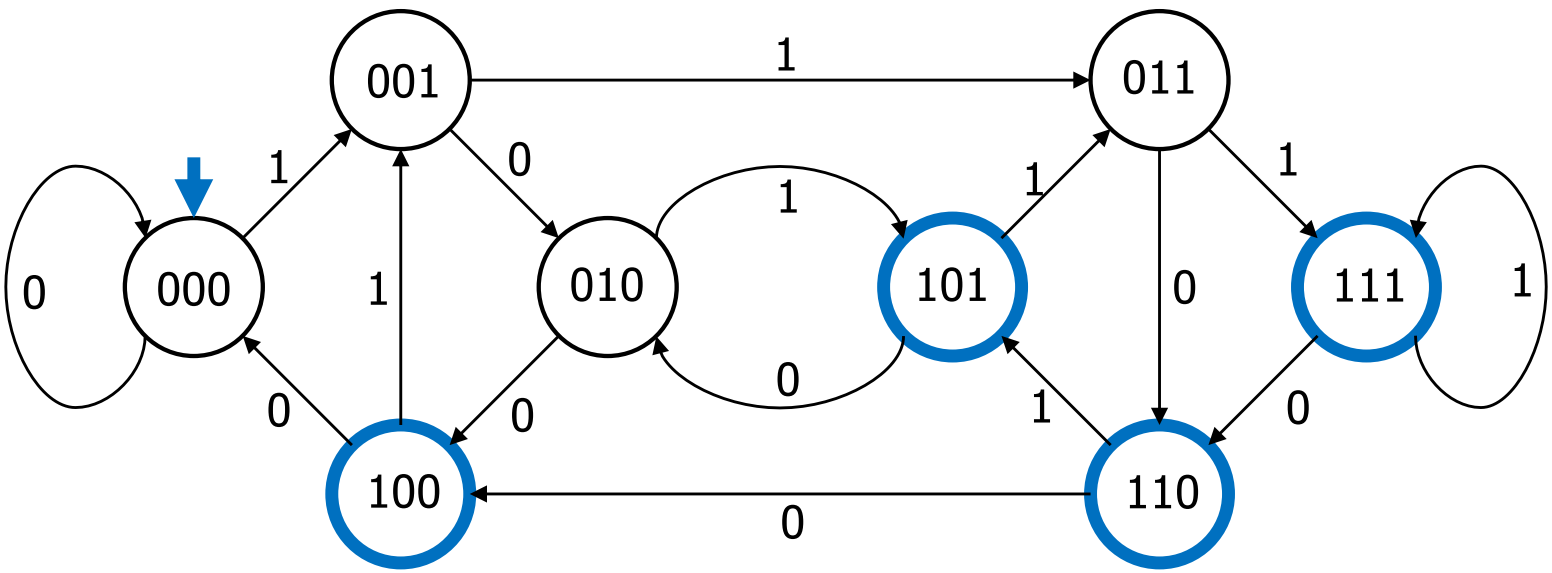


Input string 0101100



Last time: Compare with the smallest DFA





NFAs and regular expressions

Theorem: For any set of strings (language) A described by a regular expression, there is an NFA that recognizes A.

Proof idea: Structural induction based on the recursive definition of regular expressions...

Regular Expressions over Σ

- **Basis:**
  - $\varnothing$, ε are regular expressions
  - ***a*** is a regular expression for any *a* $\in \Sigma$
- **Recursive step:**
  - If **A** and **B** are regular expressions then so are:

    A $\cup$ B

    AB

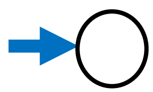
    A*

Base Case

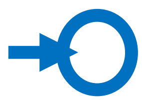
- Case $\emptyset$:
- Case ε:
- Case a:

Base Case

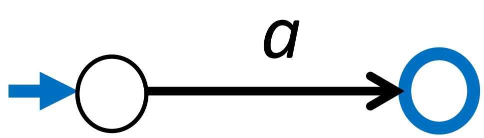
- **Case ∅:**



- **Case ε:**

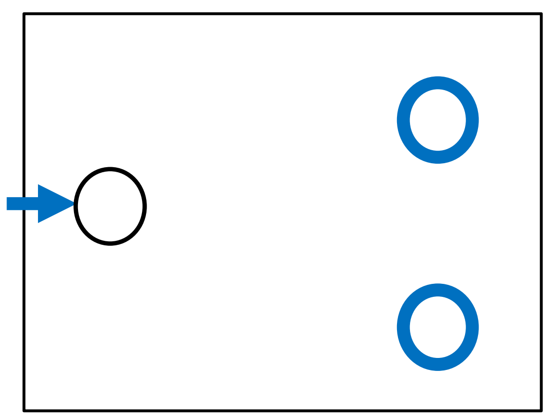


- **Case *a*:**

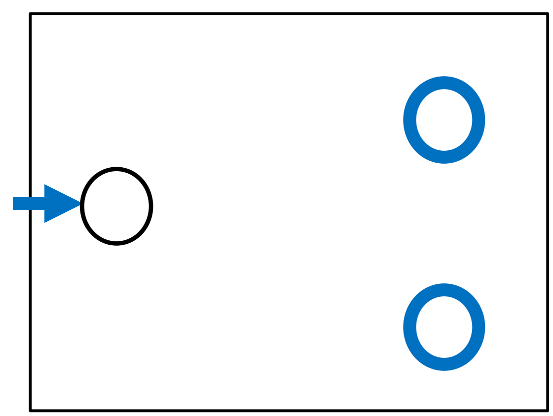


Inductive Hypothesis

- Suppose that for some regular expressions A and B there exist NFAs N_A and N_B such that N_A recognizes the language given by A and N_B recognizes the language given by B



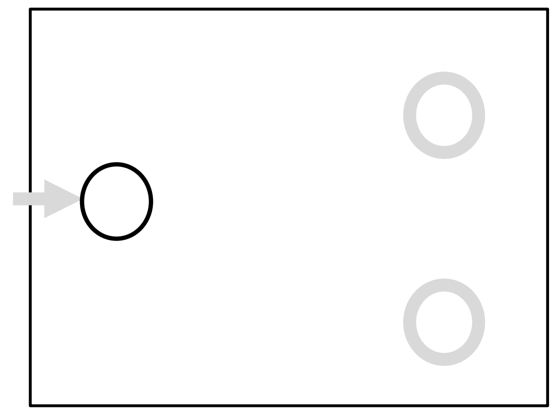
N_A



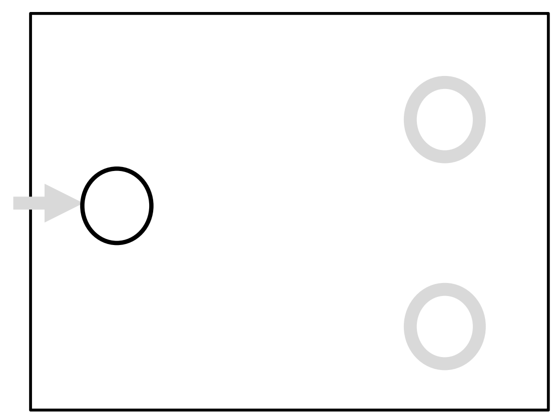
N_B

Inductive Step

Case A $\cup$ B:



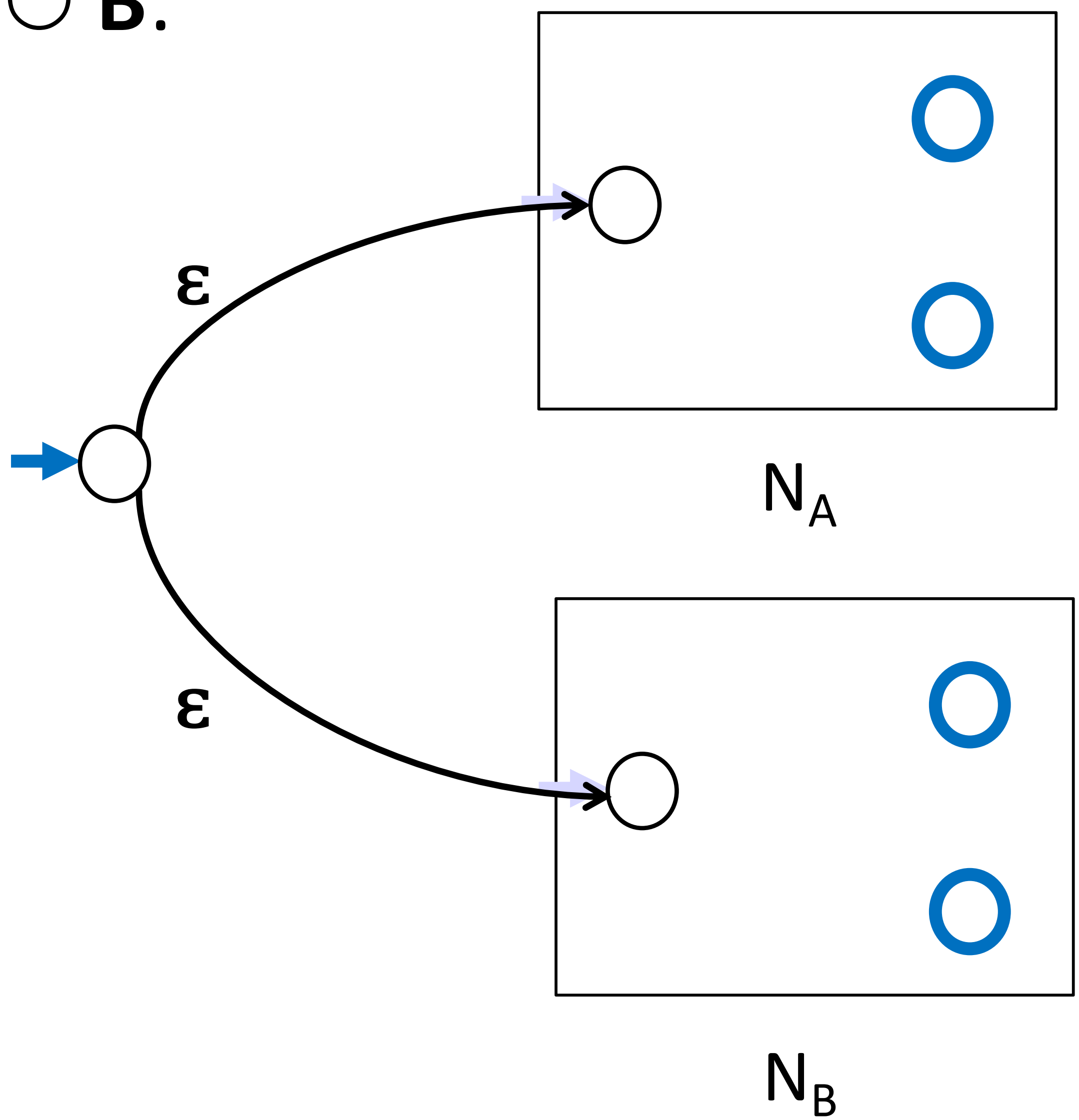
N_A



N_B

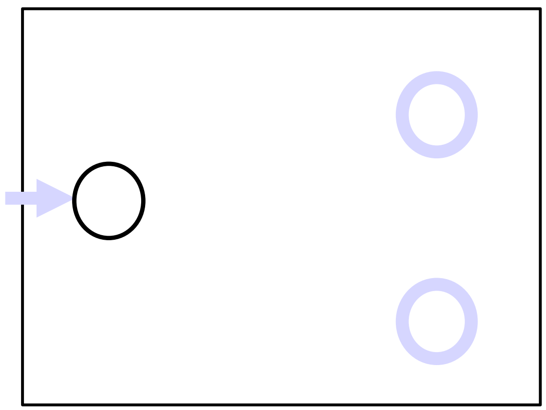
Inductive Step

Case $A \cup B$:

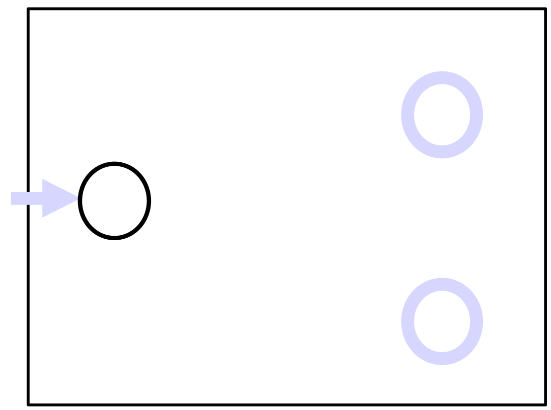


Inductive Step

Case AB:



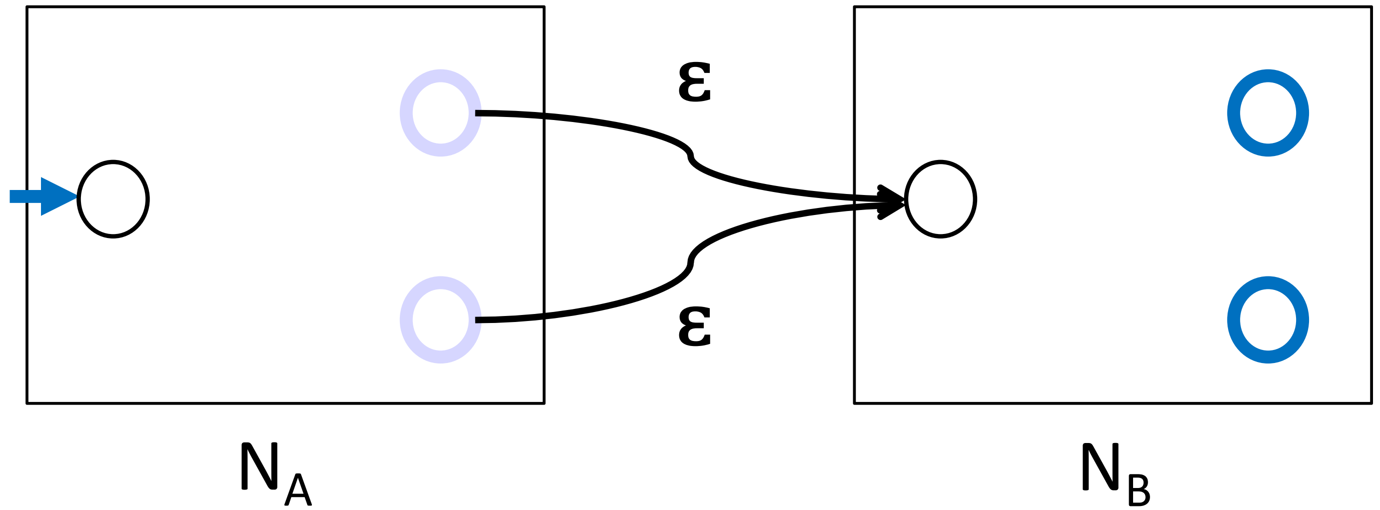
N_A



N_B

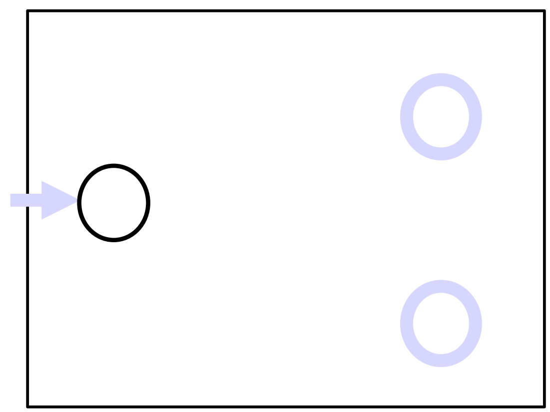
Inductive Step

Case AB:



Inductive Step

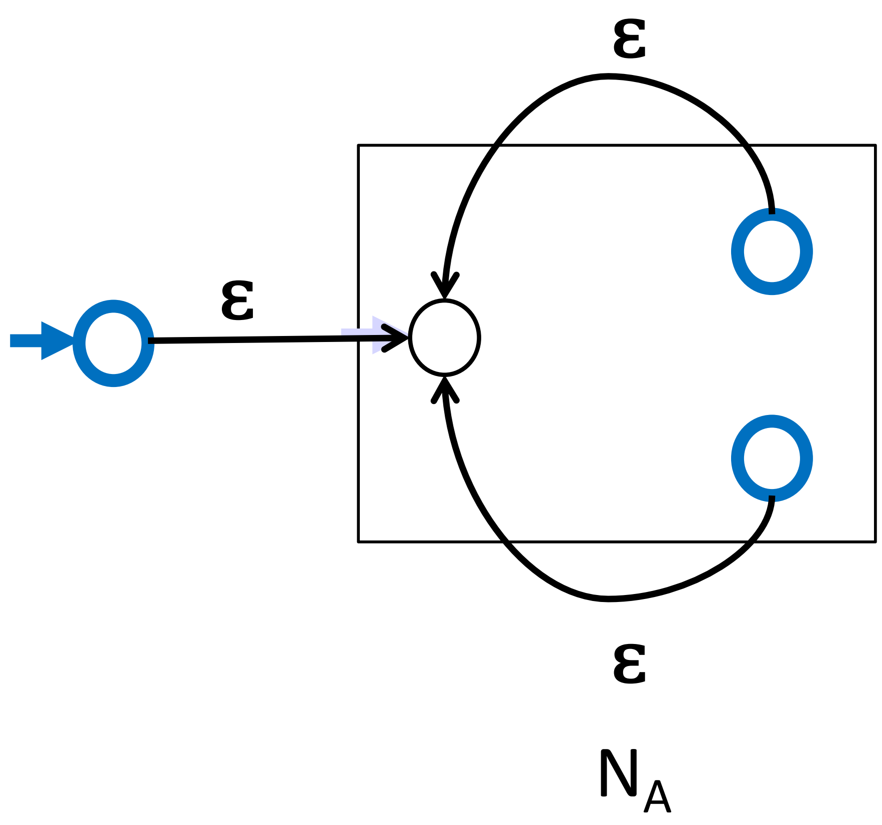
Case A*



N_A

Inductive Step

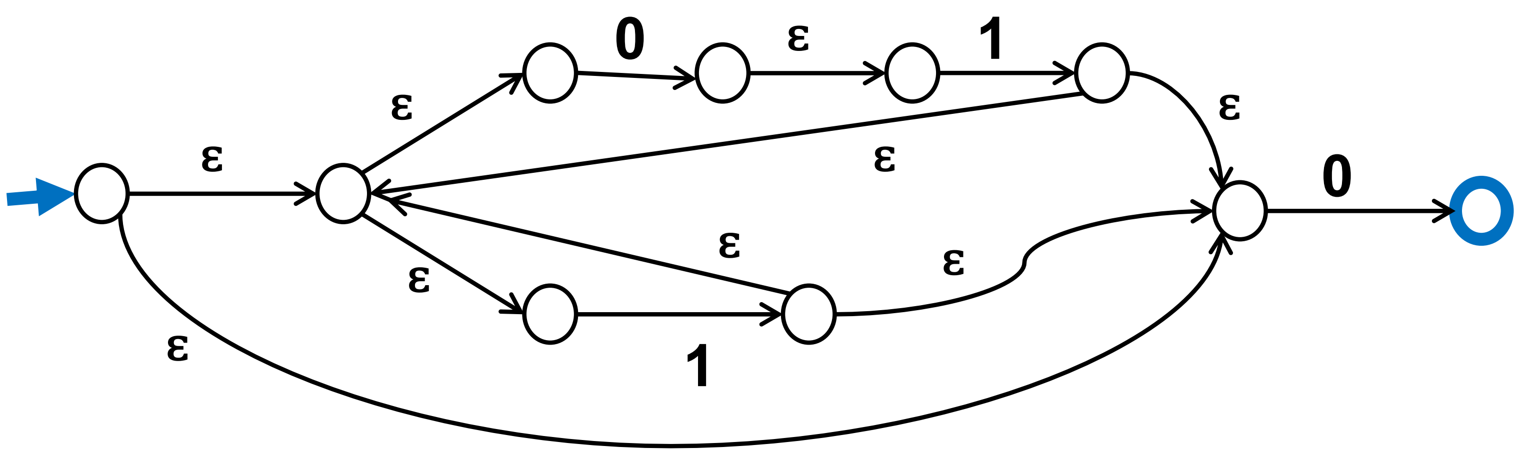
Case A*



Build an NFA for $(01 \cup 1)^*0$

Solution

(01 ∪1)*0



NFAs and DFAs

Every DFA **is** an NFA

- DFAs have requirements that NFAs don't have

Can NFAs recognize more languages?

NFAs and DFAs

Every DFA **is** an NFA

– DFAs have requirements that NFAs don't have

Can NFAs recognize more languages? No!

Theorem: For every NFA there is a DFA that recognizes exactly the same language

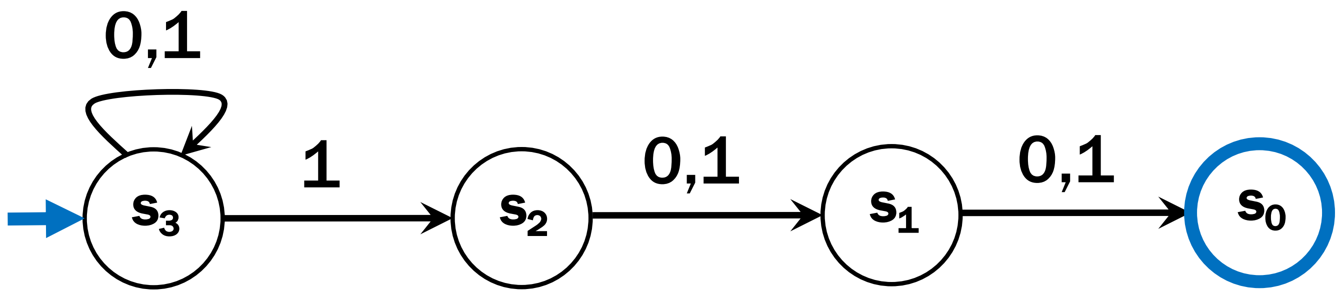
Three ways of thinking about NFAs

- Outside observer: Is there a path labeled by x from the start state to some final state?
- Perfect guesser: The NFA has input x and whenever there is a choice of what to do it magically guesses a good one (if one exists)
- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

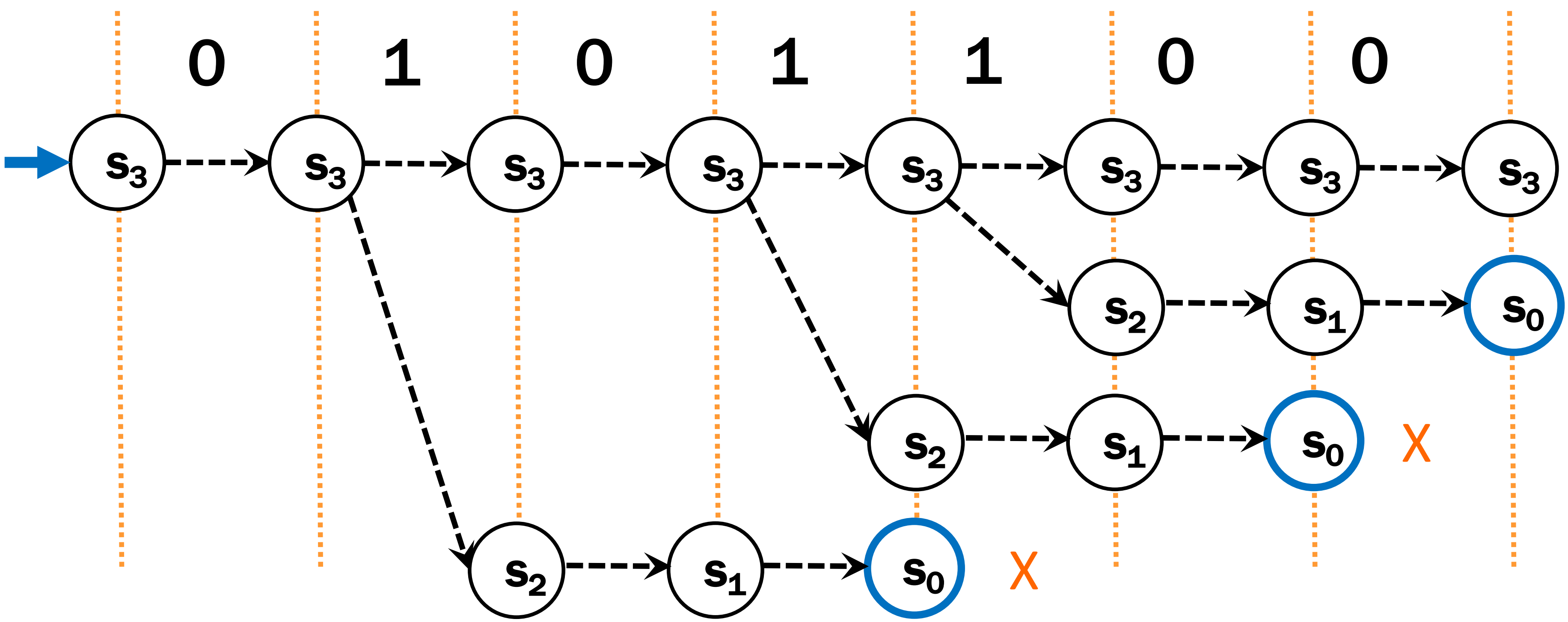
Conversion of NFAs to a DFAs

- Proof Idea:
  - The DFA keeps track of ALL the states that the part of the input string read so far can reach in the NFA
  - There will be one state in the DFA for each *subset* of states of the NFA that can be reached by some string

Parallel Exploration view of an NFA



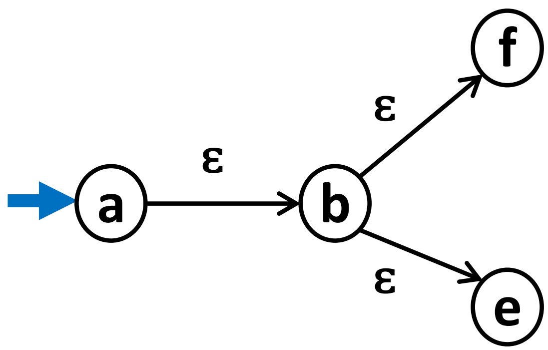
Input string 0101100



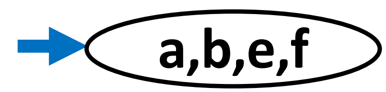
Conversion of NFAs to a DFAs

New start state for DFA

- The set of all states reachable from the start state of the NFA using only edges labeled ε



NFA

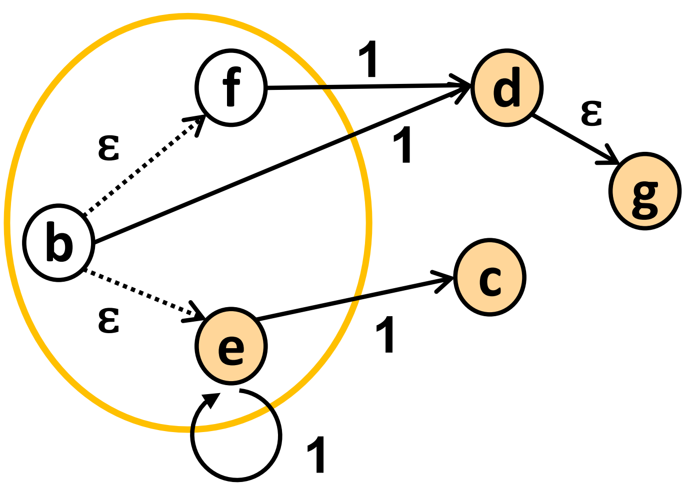


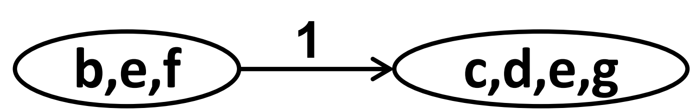
DFA

Conversion of NFAs to a DFAs

For each state of the DFA corresponding to a set S of states of the NFA and each symbol s

- **Add an edge labeled s to state corresponding to T, the set of states of the NFA reached by**
  - starting from some state in S, then
  - following one edge labeled by s, and
    then following some number of edges labeled by **ε**
- T **will be ∅ if no edges from** S **labeled** s **exist**

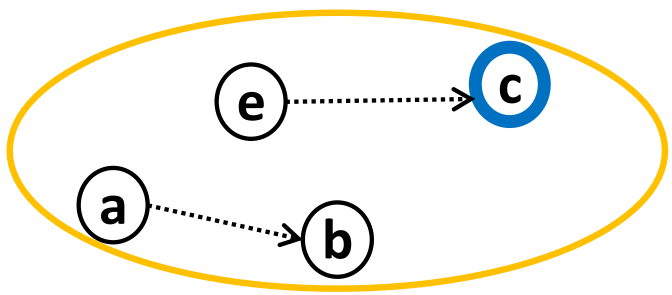




Conversion of NFAs to a DFAs

Final states for the DFA

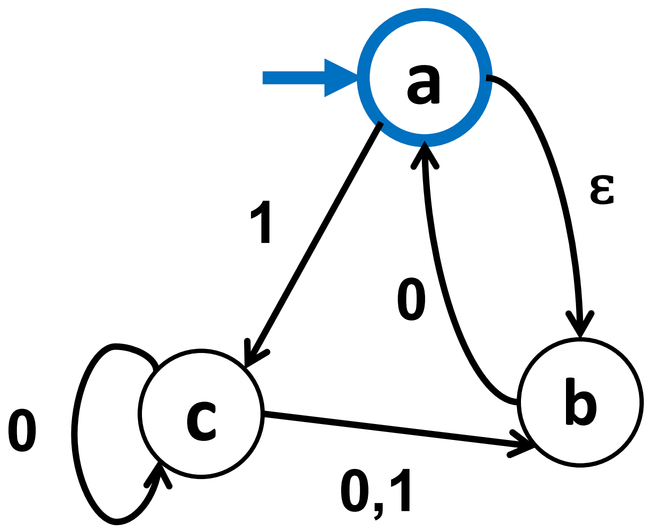
– All states whose set contain some final state of the NFA



NFA

DFA

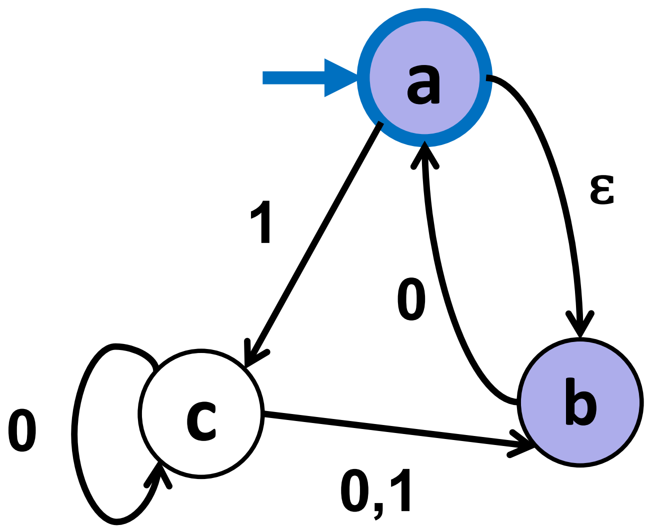
Example: NFA to DFA



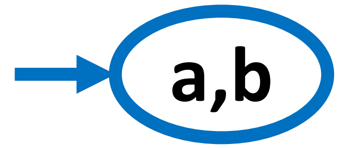
NFA

DFA

Example: NFA to DFA

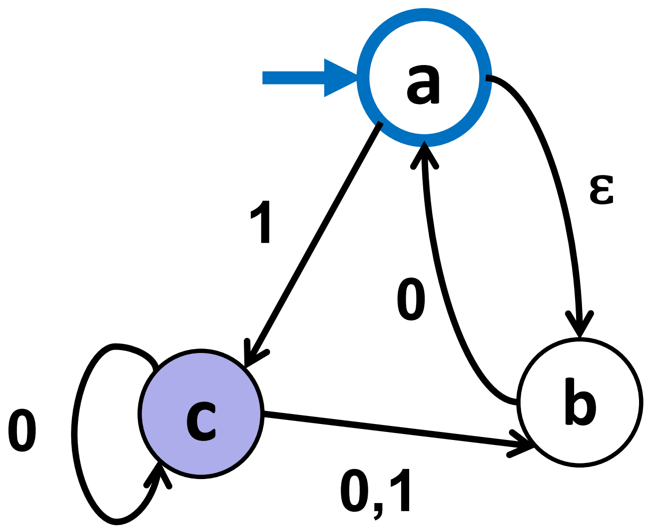


NFA

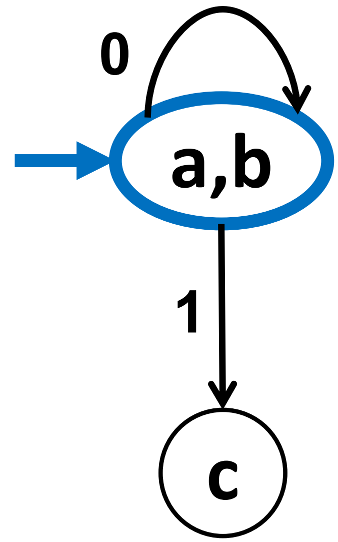


DFA

Example: NFA to DFA

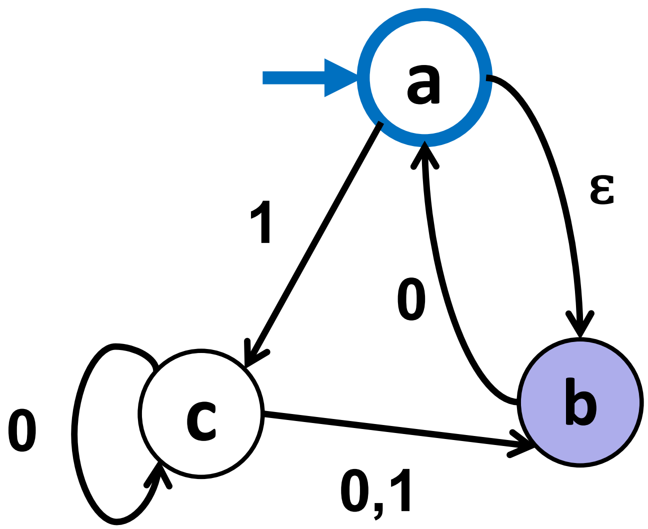


NFA

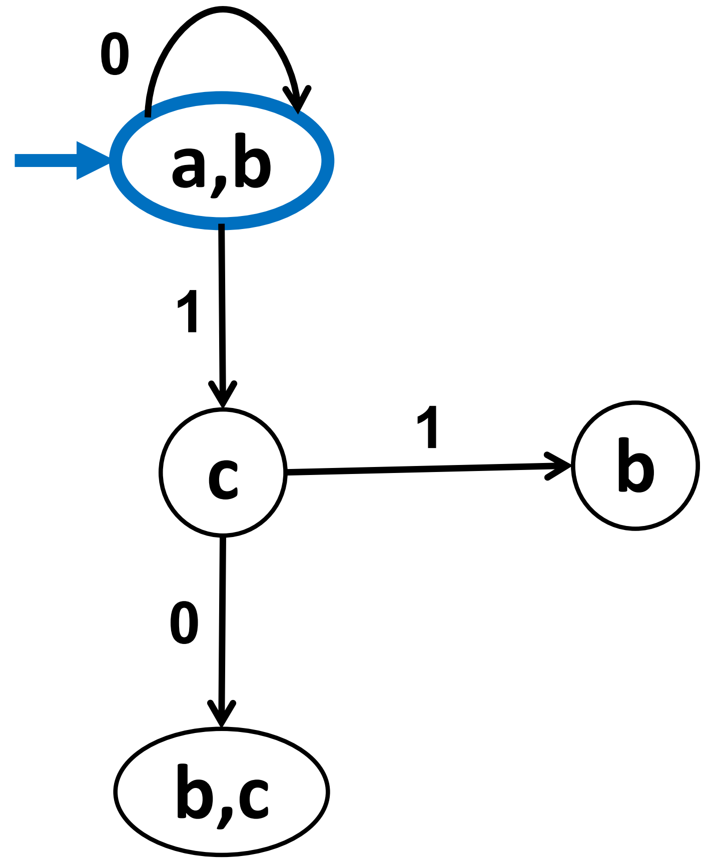


DFA

Example: NFA to DFA

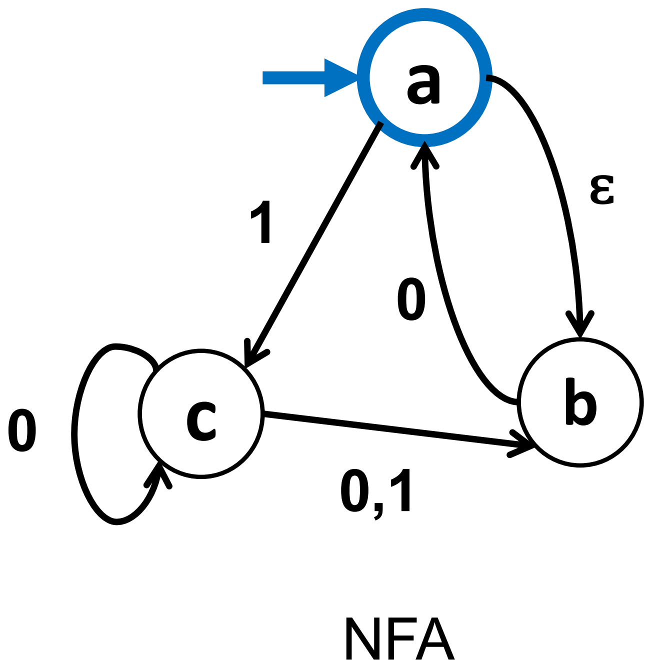


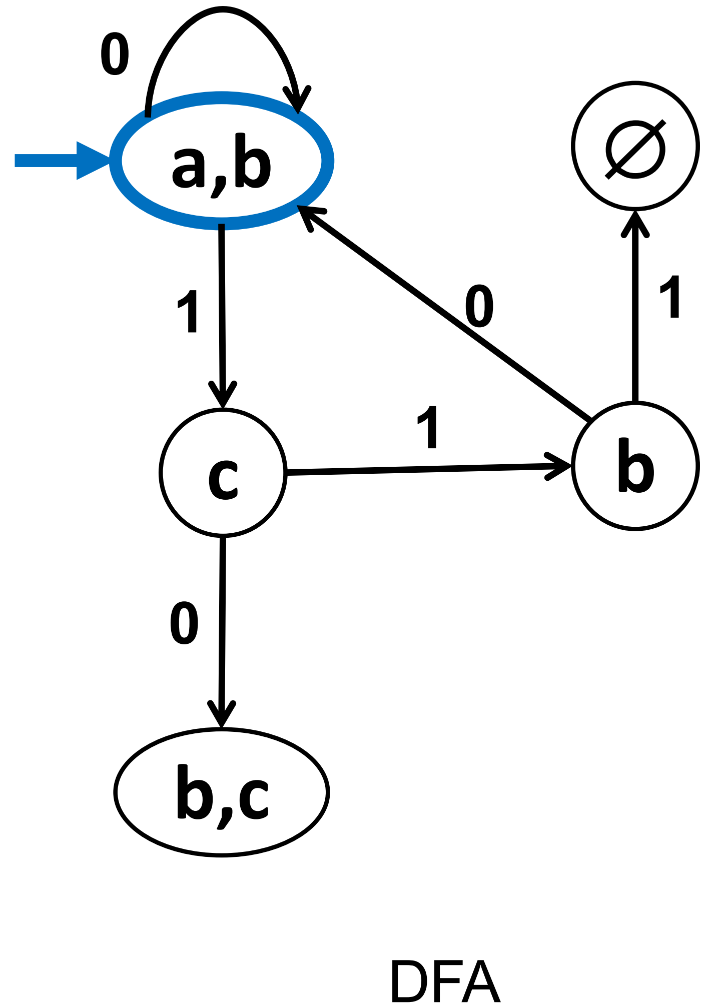
NFA



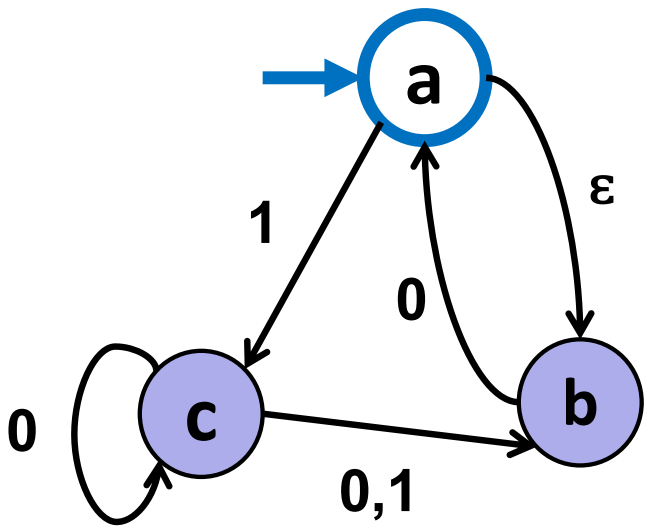
DFA

Example: NFA to DFA

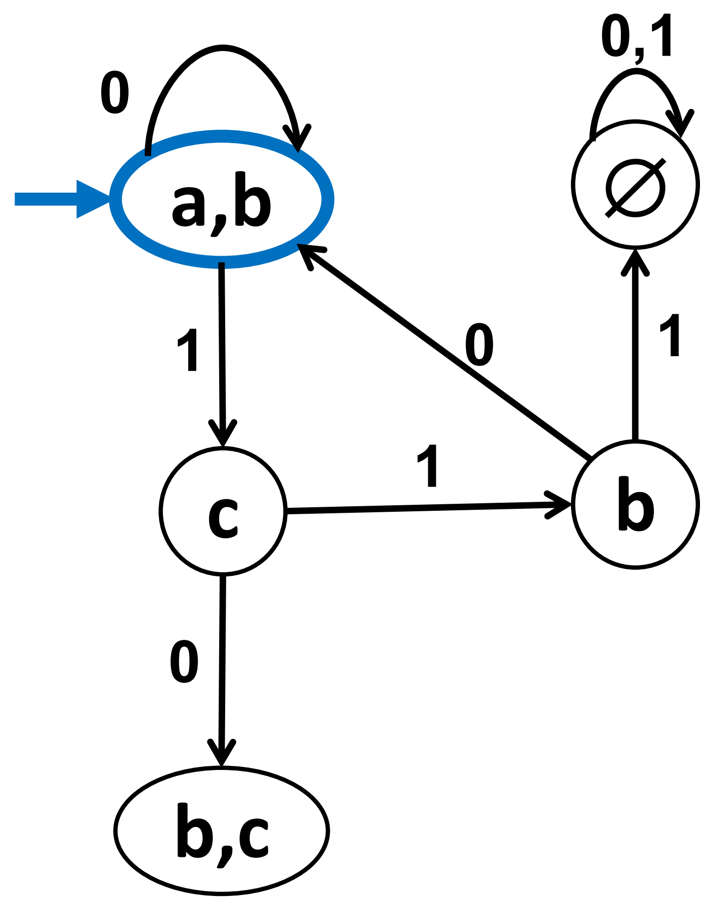




Example: NFA to DFA

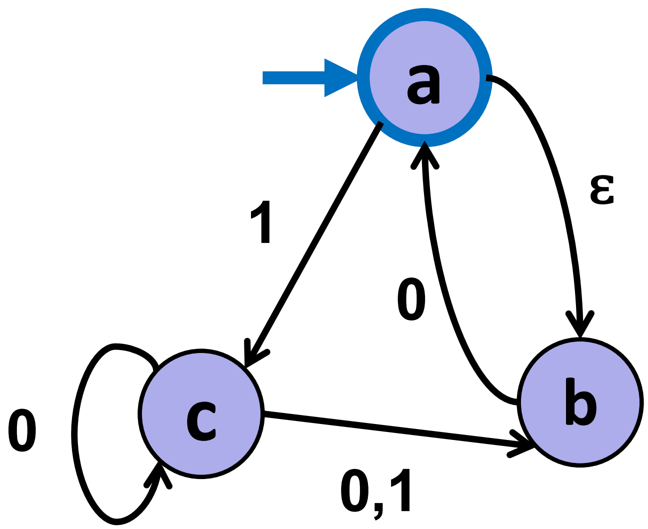


NFA

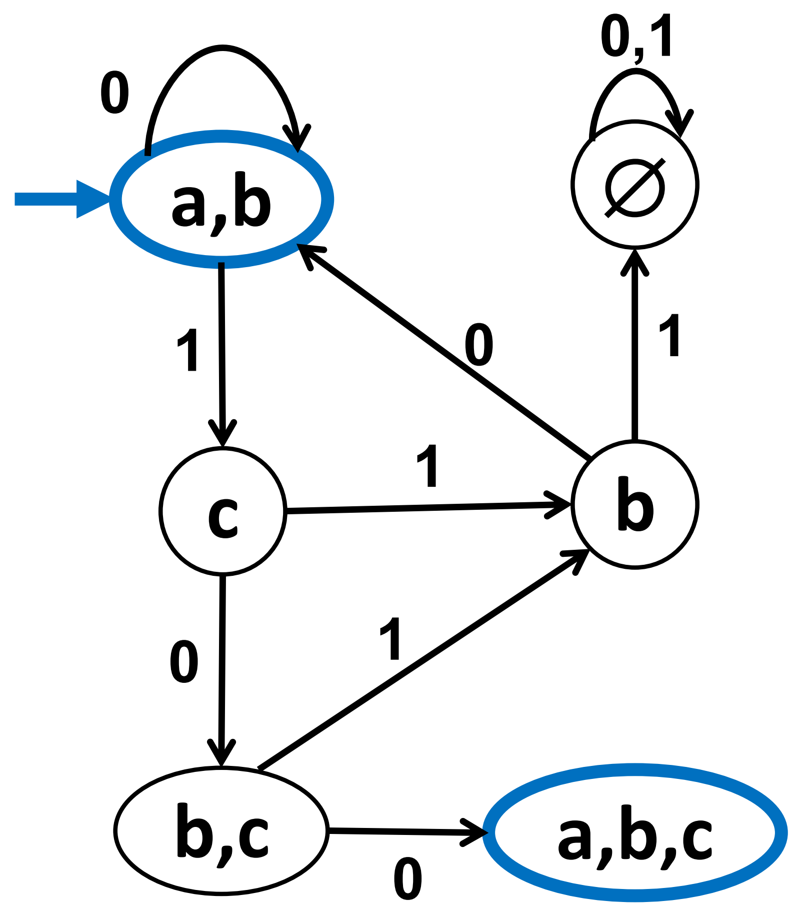


DFA

Example: NFA to DFA

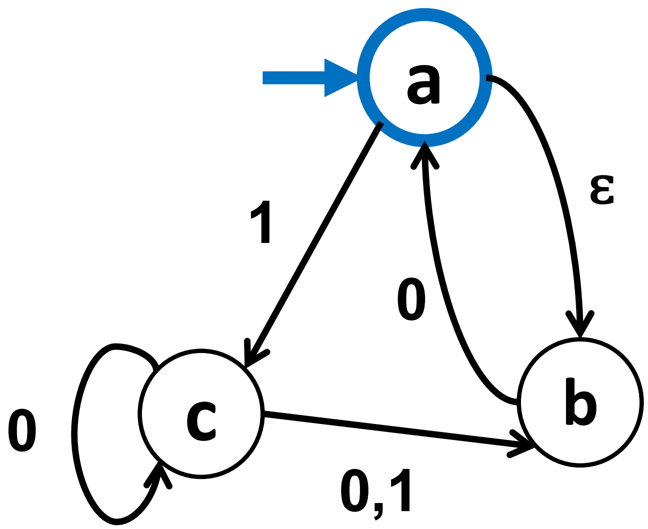


NFA

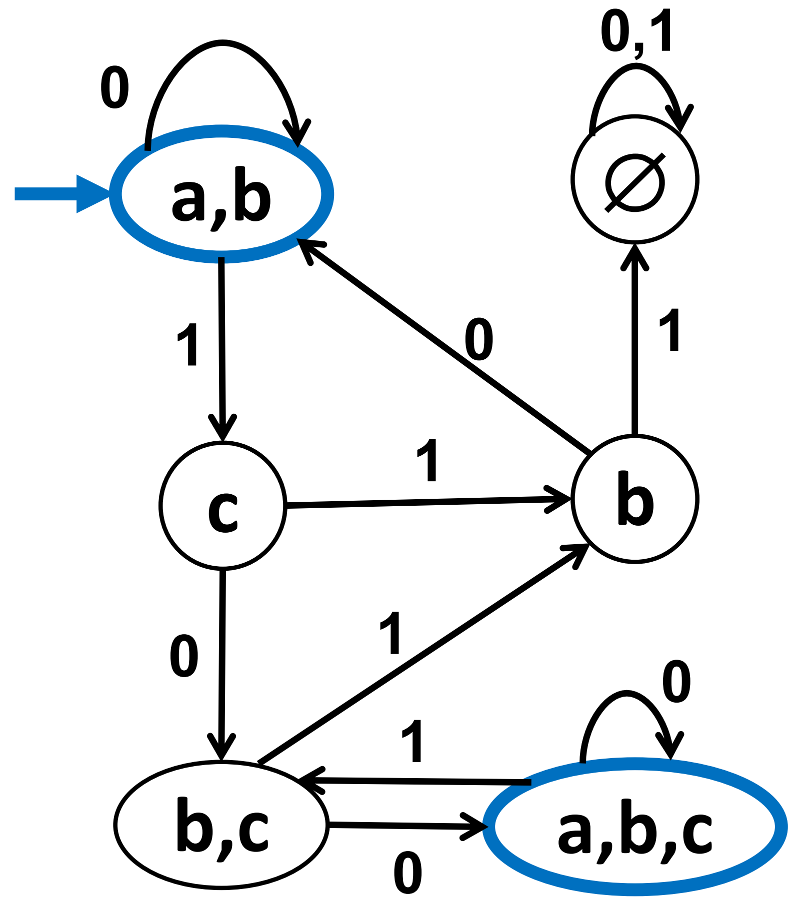


DFA

Example: NFA to DFA



NFA



DFA

Exponential Blow-up in Simulating Nondeterminism

- In general the DFA might need a state for every subset of states of the NFA
  - Power set of the set of states of the NFA
  - n-state NFA yields DFA with at most 2^n states
  - We saw an example where roughly 2^n is necessary

    "Is the n^{th} char from the end a 1?"

- The famous "P=NP?" question asks whether a similar blow-up is always necessary to get rid of nondeterminism for polynomial-time algorithms

Regular expressions $\subseteq$ NFAs $\equiv$ DFAs

We have shown how to build an optimal DFA for every regular expression

- Build NFA
- Convert NFA to DFA using subset construction
- Minimize resulting DFA

Regular expressions ≡ NFAs ≡ DFAs

We have shown how to build an optimal DFA for every regular expression

- Build NFA
- Convert NFA to DFA using subset construction
- Minimize resulting DFA

Theorem: A language is recognized by a DFA (or NFA) if and only if it has a regular expression

You need to know this fact but we won't ask you anything about the "only if" direction from DFA/NFA to regular expression. For fun, we sketch the idea.

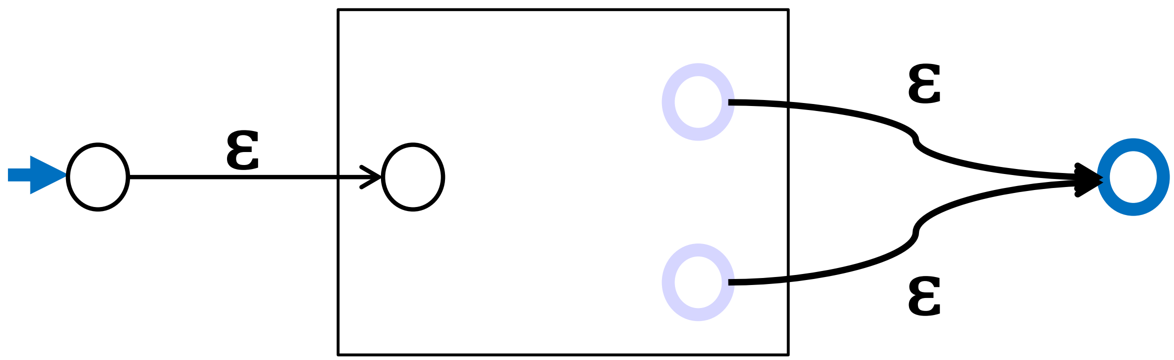
Generalized NFAs

- Like NFAs but allow
  - Parallel edges
  - Regular Expressions as edge labels

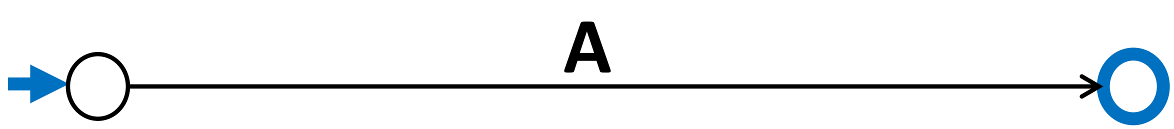
    NFAs already have edges labeled **ε** or ***a***
- An edge labeled by **A** can be followed by reading a string of input chars that is in the language represented by **A**
- Defn: A string x is accepted iff there is a *path* from start to final state *labeled by a regular expression* whose language contains x

Starting from an NFA

Add new start state and final state



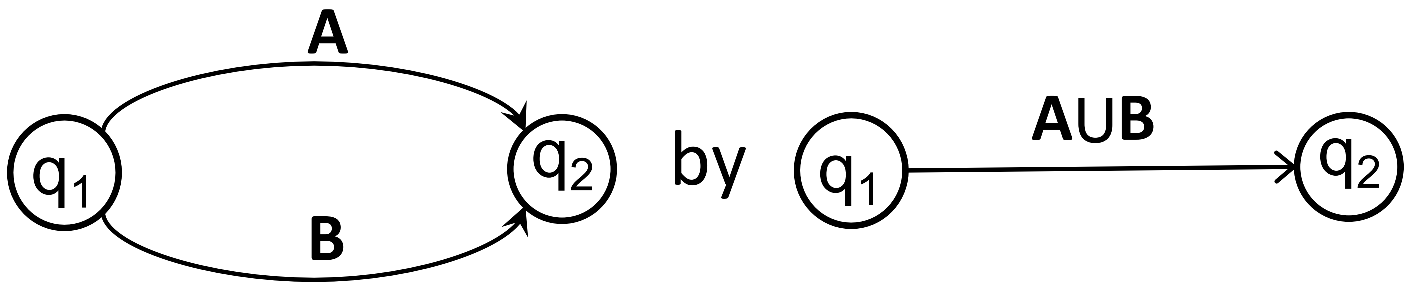
Then eliminate original states one by one, keeping the same language, until it looks like:



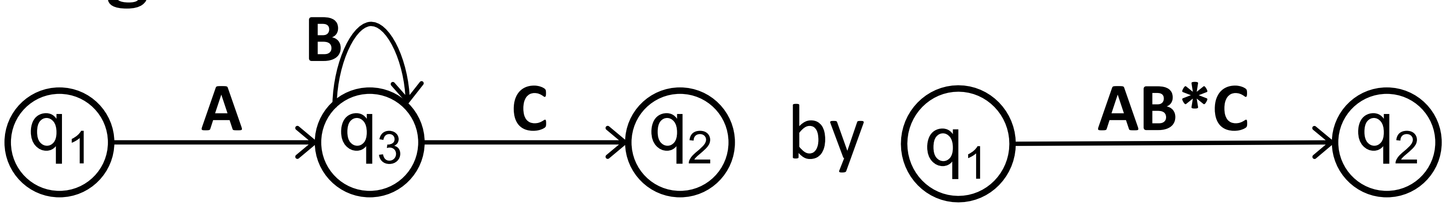
Final regular expression will be **A**

Only two simplification rules

- **Rule 1**: For any two states q_1 and q_2 with parallel edges (possibly q_1=q_2), replace



- **Rule 2**: Eliminate non-start/final state q_3 by replacing all

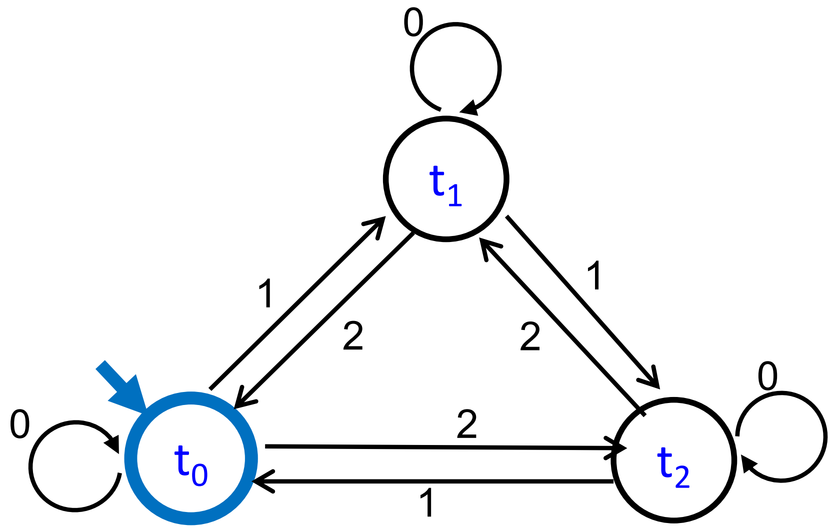


for *every* pair of states q_1, q_2 (even if q_1=q_2)

Converting an NFA to a regular expression

Consider the DFA for the mod 3 sum

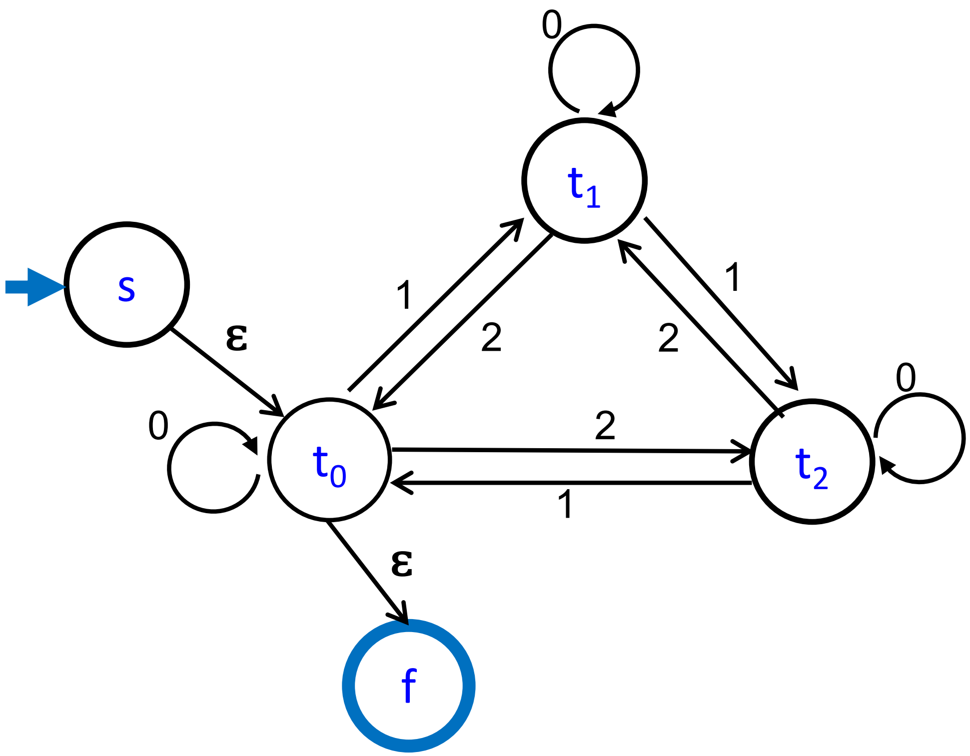
- Accept strings from {0,1,2}* where the digits mod 3 sum of the digits is 0



Splicing out a state t_1

Regular expressions to add to edges

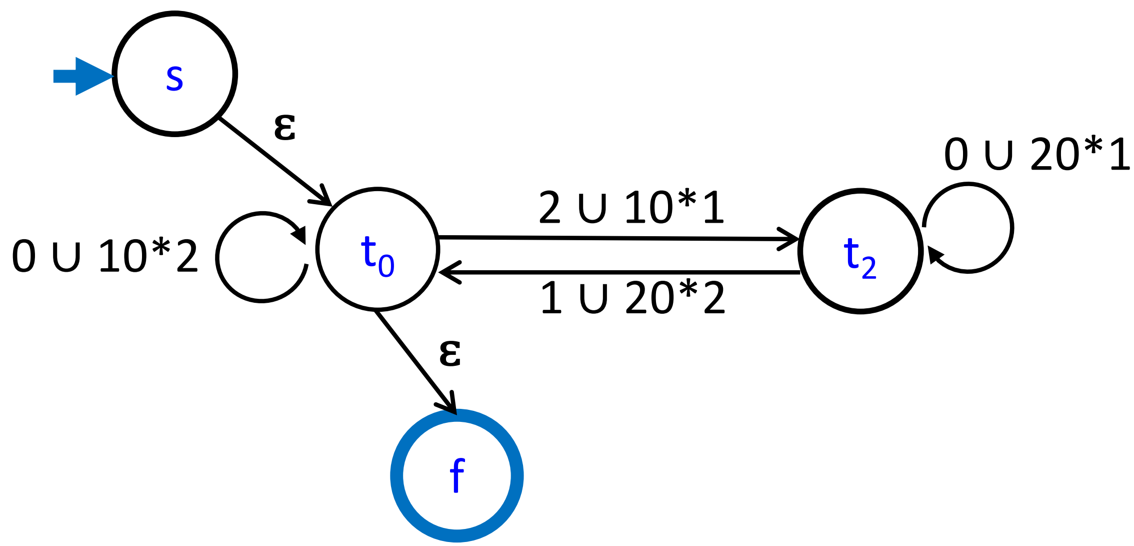
$t_0 \rightarrow t_1 \rightarrow t_0$: 10*2
$t_0 \rightarrow t_1 \rightarrow t_2$: 10*1
$t_2 \rightarrow t_1 \rightarrow t_0$: 20*2
$t_2 \rightarrow t_1 \rightarrow t_2$: 20*1



Splicing out a state t_1

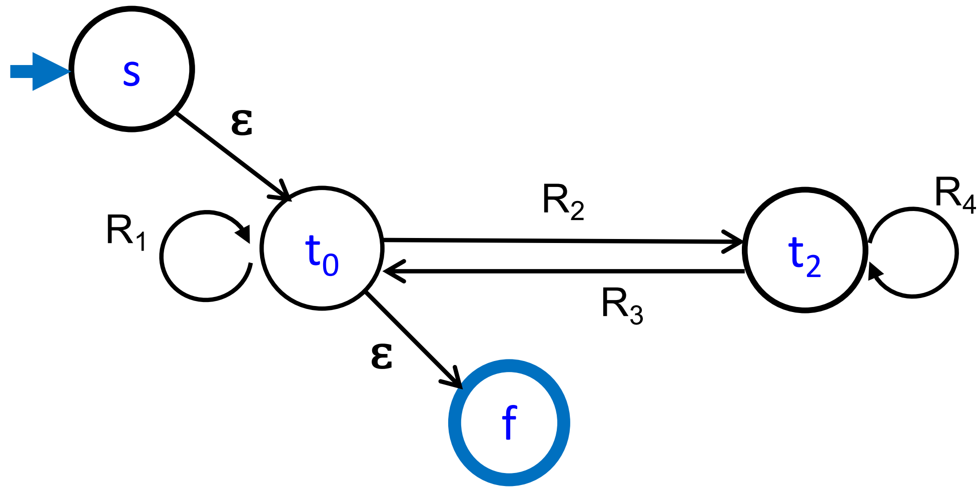
Regular expressions to add to edges

$t_0 \rightarrow t_1 \rightarrow t_0$: 10*2
$t_0 \rightarrow t_1 \rightarrow t_2$: 10*1
$t_2 \rightarrow t_1 \rightarrow t_0$: 20*2
$t_2 \rightarrow t_1 \rightarrow t_2$: 20*1

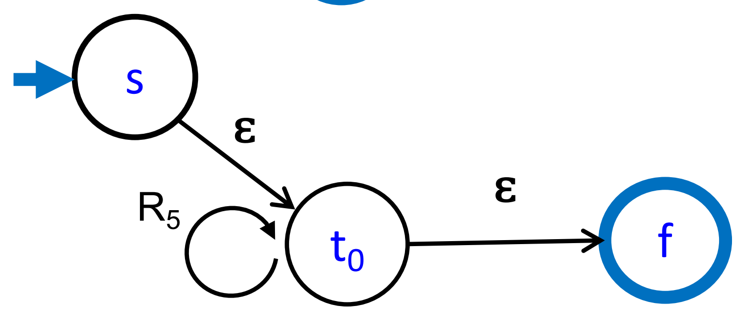


Splicing out state t_2 (and then t_0)

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$



R_5: $R_1 \cup R_2R_4{}^*R_3$



Final regular expression: $R_5{}^*=$
$(0 \cup 10^*2 \cup (2 \cup 10^*1)(0 \cup 20^*1)^*(1 \cup 20^*2))^*$