CSE 311: Foundations of Computing

Lecture 18: Structural Induction, Regular expressions

Last time: Recursive Definitions of Sets

Recursive definition

- ***Basis step:*** Some specific elements are in S
- ***Recursive step:*** Given some existing named elements in S some new objects constructed from these named elements are also in S.
- ***Exclusion rule***: Every element in S follows from the basis step and a finite number of recursive steps

Last time: Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Last time: Strings

- An *alphabet* Σ is any finite set of characters

- Set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma^*$ (ε is the empty string w/ no chars)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Functions on Recursively Defined Sets (on Σ^*)

Length:

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = a \bullet w^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$

Number of c's in a string:

$\#_c(\varepsilon) = 0$

$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

Claim: len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x ∈ Σ^*".
We prove P(y) **for all** y ∈ Σ^* **by structural induction.**

Claim: $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ **for all** $x, y \in \Sigma^*$

Let $P(y)$ **be** "$\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$" .
We prove $P(y)$ **for all** $y \in \Sigma^*$ **by structural induction.**

Base Case: $y = \varepsilon$. **For any** $x \in \Sigma^*$, $\text{len}(x \bullet \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ **since** $\text{len}(\varepsilon)=0$. **Therefore** $P(\varepsilon)$ **is true**

Claim: len(x•y) = len(x) + len(y) for all x,y $\in \Sigma^*$

Let P(y) **be** “len(x•y) = len(x) + len(y) for all x ∈ Σ^*” .
We prove P(y) **for all** y ∈ Σ^* **by structural induction.**

Base Case: y= ε. **For any** x ∈ Σ^*, len(x • ε) = len(x) = len(x) + len(ε) **since** len(ε)=0. **Therefore** P(ε) **is true**

Inductive Hypothesis: **Assume that** P(w) **is true for some arbitrary** w ∈ Σ^*

Inductive Step: **Goal: Show that** P(wa) **is true for every** a ∈ Σ

Claim: len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x $\in \Sigma^*$".
We prove P(y) **for all** y $\in \Sigma^*$ **by structural induction.**

Base Case: y= ε. **For any** x $\in \Sigma^*$, len(x • ε) = len(x) = len(x) + len(ε)
since len(ε)=0. **Therefore** P(ε) **is true**

Inductive Hypothesis: Assume that P(w) **is true for some arbitrary** w $\in \Sigma^*$

Inductive Step: **Goal: Show that** P(wa) **is true for every** a $\in \Sigma$

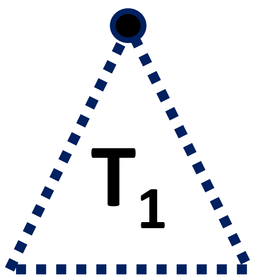
Let a $\in \Sigma$**. Let** x $\in \Sigma^*$. **Then** len(x•wa) = len((x•w)a) **by defn of •**
= len(x•w)+1 **by defn of len**
= len(x)+len(w)+1 **by I.H.**
= len(x)+len(wa) **by defn of len**

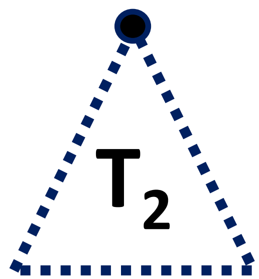
Therefore len(x•wa)= len(x)+len(wa) **for all** x $\in \Sigma^*$, **so** P(wa) **is true.**

So, by induction len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

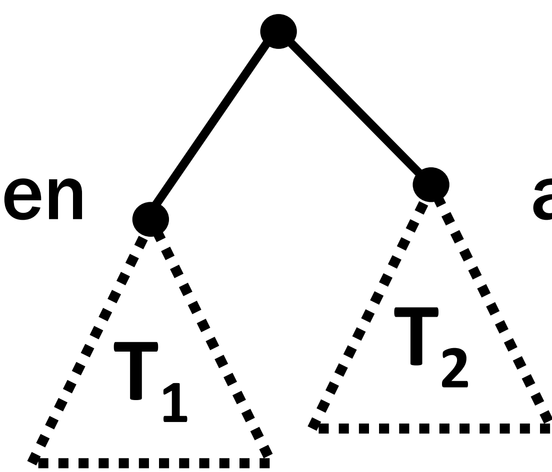
Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If 

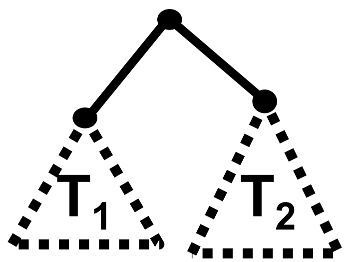
and 

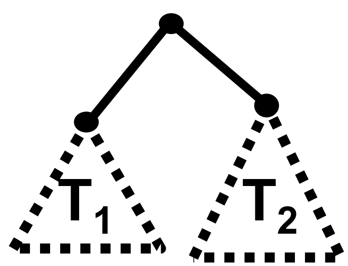
are rooted binary trees,

then 

also is a rooted binary tree.

Defining Functions on Rooted Binary Trees

- size(•) = 1
- size (

) = 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)
- height(•) = 0
- height (

)=1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

Claim: For every rooted binary tree $\mathbf{T}$, $\text{size}(\mathbf{T}) \leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

1. Let $P(\mathbf{T})$ be "$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$". We prove $P(\mathbf{T})$ for all rooted binary trees **T** by structural induction.

Claim: For every rooted binary tree T, $\text{size}(\mathbf{T}) \le 2^{\text{height}(\mathbf{T})+1} - 1$

1. **Let** $P(\mathbf{T})$ **be** "$\text{size}(T) \le 2^{\text{height}(T)+1} - 1$". **We prove** $P(\mathbf{T})$ **for all rooted binary trees T by structural induction.**
2. **Base Case:** $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ **and** $2^{0+1} - 1 = 2^1 - 1 = 2-1 = 1$ **so** $P(\bullet)$ **is true**

Claim: **For every rooted binary tree T,** $\text{size}(\mathbf{T}) \le 2^{\text{height}(\mathbf{T})+1} - 1$

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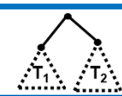
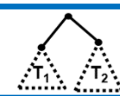
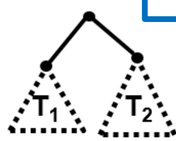
1. **Let** $P(\mathbf{T})$ **be** "$\text{size}(T) \le 2^{\text{height}(T)+1} - 1$". **We prove** $P(\mathbf{T})$ **for all rooted binary trees T by structural induction.**
2. **Base Case:** $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ **and** $1=2^1-1=2^{0+1}-1$ **so** $P(\bullet)$ **is true.**
3. **Inductive Hypothesis: Suppose that** $P(\mathbf{T_1})$ **and** $P(\mathbf{T_2})$ **are true for some rooted binary trees** $\mathbf{T_1}$ **and** $\mathbf{T_2}$.
4. **Inductive Step:**

Goal: Prove P(

).

Claim: For every rooted binary tree T, size(T) $\leq 2^{\text{height}(\mathbf{T})+1} - 1$

1. Let P(**T**) be "size(T) $\leq 2^{\text{height}(T)+1}-1$". We prove P(**T**) for all rooted binary trees **T** by structural induction.
2. **Base Case:** size(•)=1, height(•)=0 and $1=2^1-1=2^{0+1}-1$ so P(•) is true.
3. **Inductive Hypothesis:** Suppose that P($\mathbf{T_1}$) and P($\mathbf{T_2}$) are true for some rooted binary trees $\mathbf{T_1}$ and $\mathbf{T_2}$.
4. **Inductive Step:** Goal: Prove P().

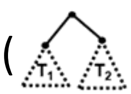
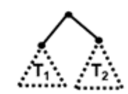
   By defn, size() $=1+\text{size}(\mathbf{T_1})+\text{size}(\mathbf{T_2})$

   $\leq 1+2^{\text{height}(\mathbf{T_1})+1}-1+2^{\text{height}(\mathbf{T_2})+1}-1$

   by IH for $\mathbf{T_1}$ and $\mathbf{T_2}$

   $= 2^{\text{height}(\mathbf{T_1})+1}+2^{\text{height}(\mathbf{T_2})+1}-1$

   $\leq 2(2^{\max(\text{height}(\mathbf{T_1}),\text{height}(\mathbf{T_2}))+1})-1$

   $= 2(2^{\text{height}(}$  $^{)}) - 1 = 2^{\text{height}(}$  $^{)+1} - 1$

   which is what we wanted to show.
5. So, the P(T) is true for all rooted bin. trees by structural induction.

Languages: Sets of Strings

- Sets of strings that satisfy special properties are called *languages*. Examples:
  - English sentences
  - Syntactically correct Java/C/C++ programs
  - $\Sigma^* =$ All strings over alphabet Σ
  - Palindromes over Σ
  - Binary strings that don't have a 0 after a 1
  - Legal variable names. keywords in Java/C/C++
  - Binary strings with an equal # of 0's and 1's

Regular Expressions

Regular expressions over Σ

- **Basis**:

  ε is a regular expression (could also include $\varnothing$)

  a is a regular expression for any $a \in \Sigma$

- **Recursive step**:
  - If **A** and **B** are regular expressions then so are:

    A $\cup$ **B**

    AB

    A*

Each Regular Expression is a "pattern"

- ε matches the **empty string**
- a matches the one character string a
- $A \cup B$ matches all strings that either **A** matches or **B** matches (or both)
- **AB** matches all strings that have a first part that **A** matches followed by a second part that **B** matches
- **A*** matches all strings that have any number of strings (even 0) that **A** matches, one after another

Examples

001*

0*1*

Examples

001*

{00, 001, 0011, 00111, ...}

0*1*

Any number of 0's followed by any number of 1's

Examples

$(0 \cup 1)\, 0\, (0 \cup 1)\, 0$

$(0^*1^*)^*$

Examples

$(\mathit{0} \cup \mathit{1})\ \mathit{0}\ (\mathit{0} \cup \mathit{1})\ \mathit{0}$

{0000, 0010, 1000, 1010}

$(\mathit{0}^*\mathit{1}^*)^*$

All binary strings

Examples

$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$

$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$

Examples

$(0 \cup 1)^*\ 0110\ (0 \cup 1)^*$

Binary strings that contain “0110”

$(00 \cup 11)^*\ (01010 \cup 10001)\ (0 \cup 1)^*$

Binary strings that begin with pairs of characters followed by “01010” or “10001”

Regular Expressions in Practice

- Used to define the "tokens": e.g., legal variable names, keywords in programming languages and compilers
- Used in **`grep`**, a program that does pattern matching searches in UNIX/LINUX
- Pattern matching using regular expressions is an essential feature of PHP
- We can use regular expressions in programs to process strings!

Regular Expressions in Java

- Pattern p = Pattern.compile("a*b");
- Matcher m = p.matcher("aaaaab");
- boolean b = m.matches();

`[01]`	a 0 or a 1	`^` start of string	`$` end of string
`[0-9]`	any single digit	`\.` period	`\,` comma `\-` minus
`.`	any single character		

ab	a followed by b	(**AB**)
(a|b)	a or b	(**A** $\cup$ **B**)
a**?**	zero or one of a	(**A** $\cup$ ε)
a*****	zero or more of a	**A***
a**+**	one or more of a	**AA***

- e.g. `^[\-+]?[0-9]*(\.|\,)?[0-9]+$`

  General form of decimal number e.g. 9.12 or -9,8 (Europe)