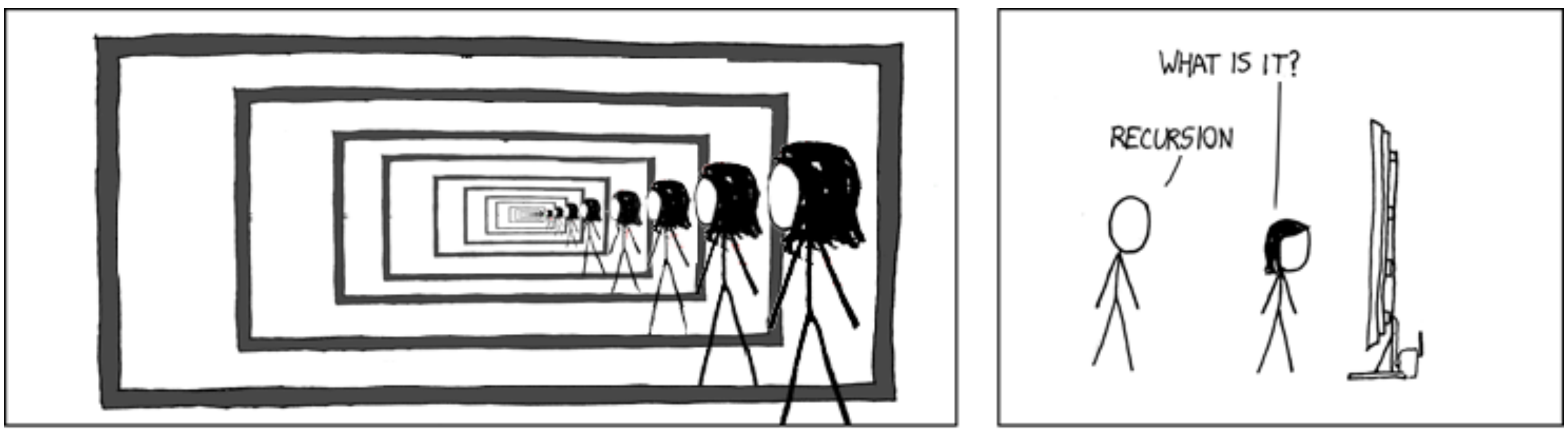
CSE 311: Foundations of Computing

Lecture 17: Recursively Defined Sets & Structural Induction



Midterm

- Monday, May 13th in class
- Closed book, closed notes
  - will include inference rules & equivalences if helpful
  - expect you remember congruence, divides, inverse, etc.
- Covers material up to end of ordinary induction.

- Practice problems & midterm on the website

- TA-led review session:

  Saturday, May 11th, 2-4 pm in SMI 120

Midterm

- 5 problems covering:
  - Logic / English translation
  - Boolean circuits, algebra, and normal forms
  - Solving modular equations
  - Induction
  - Modular arithmetic
  - Set theory
  - English proofs

Recursive Definitions of Sets

Natural numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+2 \in S$

Recursive Definition of Sets

Recursive definition of set S

- **Basis Step:** $0 \in S$
- **Recursive Step: If** $x \in S$**, then** $x + 2 \in S$
- **Exclusion Rule: Every element in** S **follows from the basis step and a finite number of recursive steps.**

We need the exclusion rule because otherwise $S=\mathbb{N}$ would satisfy the other two parts. However, we won't always write it down on these slides.

Recursive Definitions of Sets

Natural numbers

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Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+2 \in S$

Powers of 3:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$.

Basis: $[0, 0] \in S$, $[1, 1] \in S$

Recursive: If $[n-1, x] \in S$ and $[n, y] \in S$, then $[n+1, x + y] \in S$. ?

Recursive Definitions of Sets

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Even numbers

Basis: $0 \in S$

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Basis: $[0, 0] \in S$, $[1, 1] \in S$

Recursive: If $[n-1, x] \in S$ and $[n, y] \in S$, then $[n+1, x + y] \in S$.

Fibonacci numbers

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma^*$ (ε is the empty string w/ no chars)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same backwards and forwards

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$

All Binary Strings with no 1's before 0's

All Binary Strings with no 1's before 0's

Basis:

$\varepsilon \in S$

Recursive:

If $x \in S$, **then** $0x \in S$

If $x \in S$, **then** $x1 \in S$

Functions on Recursively Defined Sets (on Σ^*)

Length:

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = a \bullet w^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Number of c's in a string:

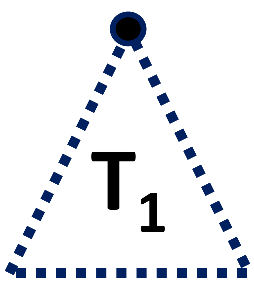
$\#_c(\varepsilon) = 0$

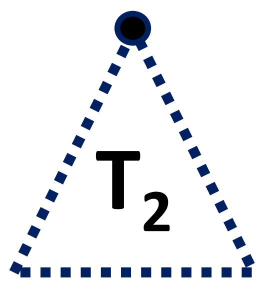
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

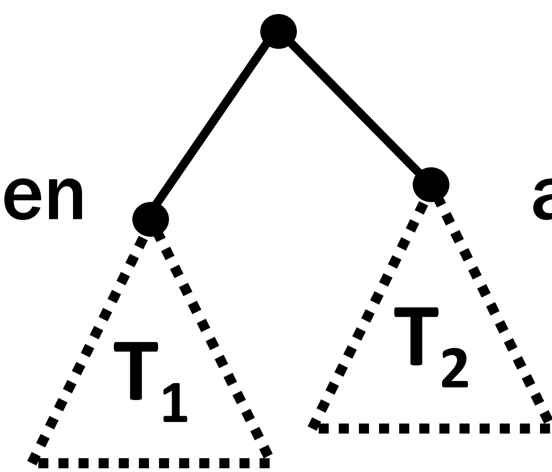
Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If 

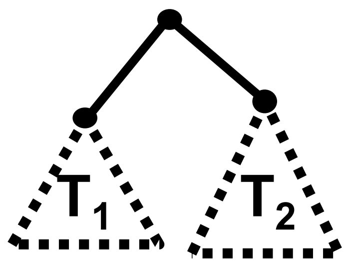
and 

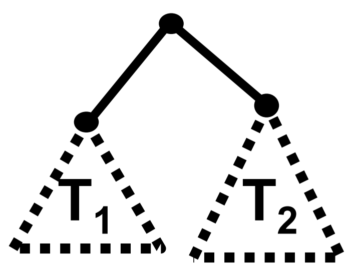
are rooted binary trees,

then 

also is a rooted binary tree.

Defining Functions on Rooted Binary Trees

- size(•) = 1
- size () $= 1 + \text{size}(\mathbf{T_1}) + \text{size}(\mathbf{T_2})$

- height(•) = 0
- height () $= 1 + \max\{\text{height}(\mathbf{T_1}), \text{height}(\mathbf{T_2})\}$

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

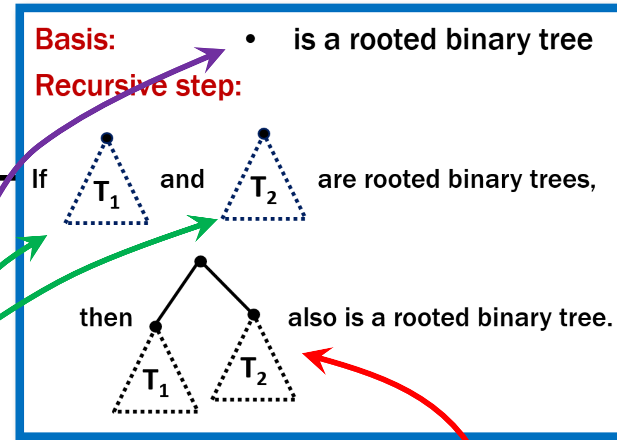
Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Structural Induction



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Conclude that $\forall\, x \in S, P(x)$

Structural Induction vs. Ordinary Induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Define $Q(n)$ to be "for all $x \in S$ that can be constructed in at most n recursive steps, $P(x)$ is true."

Using Structural Induction

- **Let S be given by…**
  - **Basis:** $6 \in S$; $15 \in S$;
  - **Recursive:** **if** $x, y \in S$ **then** $x + y \in S$.

Claim: **Every element of S is divisible by 3.**

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1. Let $P(x)$ be "$3|x$". We prove that $P(x)$ is true for all $x \in S$ by structural induction.

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: Every element of S is divisible by 3.

1. Let P(x) be "3|x". We prove that P(x) is true for all x ∈ S by structural induction.
2. Base Case: 3|6 and 3|15 so P(6) and P(15) are true

Basis: $6 \in S$; $15 \in S$;

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3. Inductive Hypothesis: Suppose that P(x) and P(y) are true for some arbitrary x,y ∈ S
4. Inductive Step: Goal: Show P(x+y)

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Claim: Every element of S is divisible by 3.

1. **Let** P(x) **be** "3|x". **We prove that** P(x) **is true for all** x ∈ S **by structural induction.**
2. **Base Case:** 3|6 **and** 3|15 **so** P(6) **and** P(15) **are true**
3. **Inductive Hypothesis: Suppose that** P(x) **and** P(y) **are true for some arbitrary** x,y ∈ S
4. **Inductive Step:** **Goal: Show** P(x+y)

   Since P(x) **is true,** 3|x **and so** x=3m **for some integer** m **and since** P(y) **is true,** 3|y **and so** y=3n **for some integer** n**.** **Therefore** x+y=3m+3n=3(m+n) **and thus** 3|(x+y).

   Hence P(x+y) **is true.**
5. **Therefore by induction** 3|x **for all** x ∈ S.

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x $\in \Sigma^*$ " .

We prove P(y) **for all** y $\in \Sigma^*$ **by structural induction.**

Claim: len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x ∈ Σ^*" .
We prove P(y) **for all** y ∈ Σ^* **by structural induction.**

Base Case (y= ε): **Let** x ∈ Σ^* **be arbitrary. Then,** len(x • ε) = len(x) = len(x) + len(ε) **since** len(ε)=0. **Since** x **was arbitrary,** P(ε) **holds.**

Claim: len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

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Inductive Hypothesis: Assume that P(w) **is true for some arbitrary** w $\in \Sigma^*$

Inductive Step: $\boxed{\text{Goal: Show that } P(wa) \text{ is true for every } a \in \Sigma}$

Claim: len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

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Inductive Step: **Goal: Show that** P(wa) **is true for every** a ∈ Σ

Let a ∈ Σ. **Let** x ∈ Σ^*. **Then** len(x•wa) = len((x•w)a) **by defn of •**
= len(x•w)+1 **by defn of len**
= len(x)+len(w)+1 **by I.H.**
= len(x)+len(wa) **by defn of len**

Therefore len(x•wa)= len(x)+len(wa) **for all** x ∈ Σ^***, so** P(wa) **is true.**

So, by induction len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

Claim: **For every rooted binary tree T,** size(**T**) $\leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

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1. Let $P(\mathbf{T})$ be "$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$". We prove $P(\mathbf{T})$ for all rooted binary trees **T** by structural induction.

Claim: For every rooted binary tree T, $\text{size}(\mathbf{T}) \le 2^{\text{height}(\mathbf{T}) + 1} - 1$

1. **Let** $P(\mathbf{T})$ **be** "$\text{size}(T) \le 2^{\text{height}(T)+1}–1$". **We prove** $P(\mathbf{T})$ **for all rooted binary trees T by structural induction.**
2. **Base Case:** $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$, **and** $2^{0+1}–1=2^1–1=1$ **so** $P(\bullet)$ **is true.**

Claim: **For every rooted binary tree T,** $\text{size}(\mathbf{T}) \leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

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1. **Let** $P(\mathbf{T})$ **be** "$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$". **We prove** $P(\mathbf{T})$ **for all rooted binary trees T by structural induction.**
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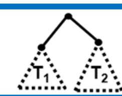
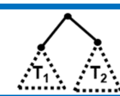
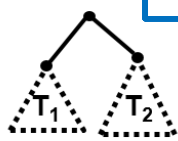
Goal: Prove P(

).

Claim: For every rooted binary tree T, size(T) $\leq 2^{\text{height}(\mathbf{T})+1} - 1$

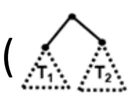
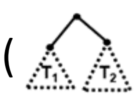
1. Let P(**T**) be "size(T) $\leq 2^{\text{height}(T)+1} - 1$". We prove P(**T**) for all rooted binary trees **T** by structural induction.
2. **Base Case:** size(•)=1, height(•)=0, and $2^{0+1}-1=2^1-1=1$ so P(•) is true.
3. **Inductive Hypothesis:** Suppose that P($\mathbf{T_1}$) and P($\mathbf{T_2}$) are true for some rooted binary trees $\mathbf{T_1}$ and $\mathbf{T_2}$.
4. **Inductive Step:** Goal: Prove P().

By defn, size() $=1+\text{size}(\mathbf{T_1})+\text{size}(\mathbf{T_2})$

$\leq 1+2^{\text{height}(\mathbf{T_1})+1}-1+2^{\text{height}(\mathbf{T_2})+1}-1$ by IH for $\mathbf{T_1}$ and $\mathbf{T_2}$

$\leq 2^{\text{height}(\mathbf{T_1})+1}+2^{\text{height}(\mathbf{T_2})+1}-1$

$\leq 2(2^{\max(\text{height}(\mathbf{T_1}),\text{height}(\mathbf{T_2}))+1})-1$

$\leq 2(2^{\text{height}($  $)})-1 \leq 2^{\text{height}($  $)+1}-1$

which is what we wanted to show.

5. So, the P(T) is true for all rooted bin. trees by structural induction.