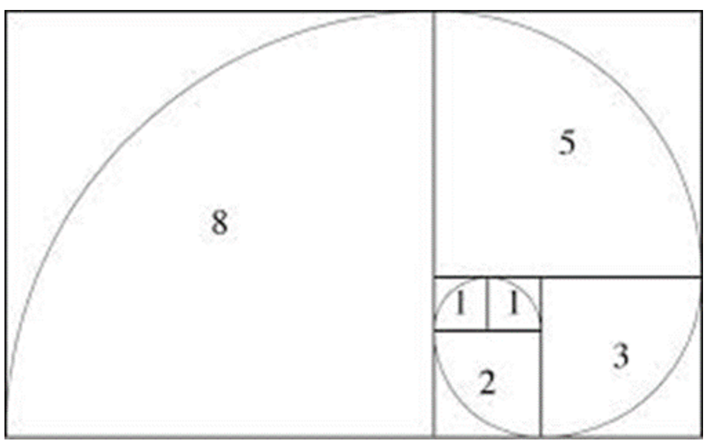
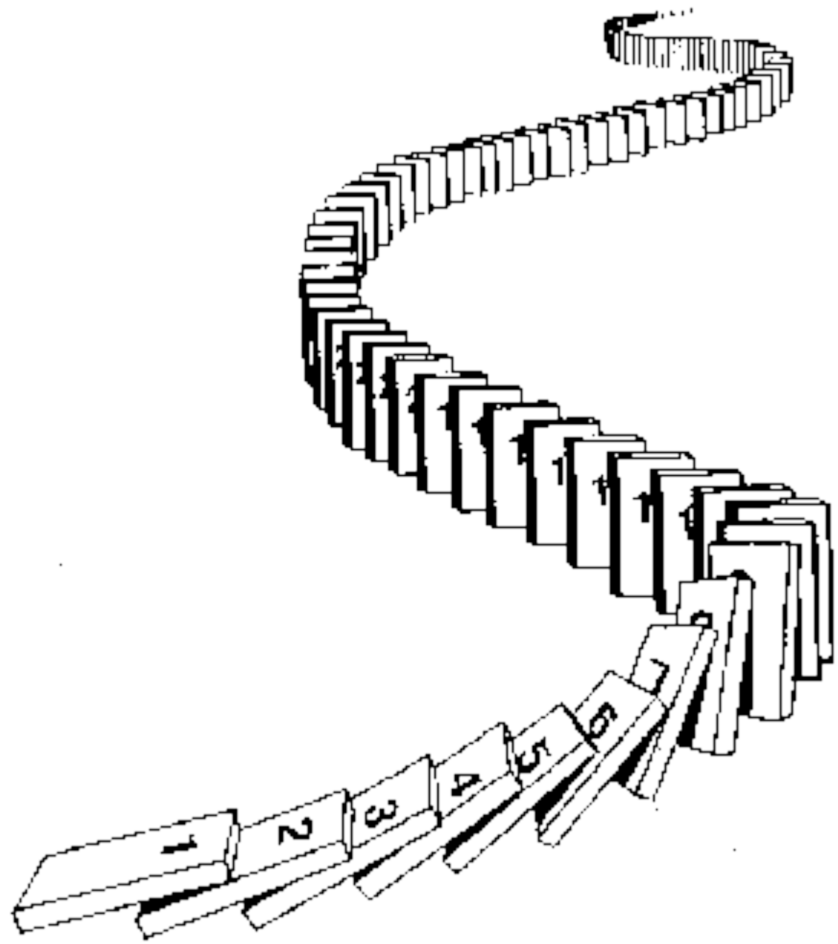
CSE 311: Foundations of Computing

Lecture 16: Recursion & Strong Induction
Applications: Fibonacci & Euclid





Last time: recursive definitions of functions

- $F(0) = 0;\ \ F(n+1) = F(n) + 1$ **for all** $n \geq\ 0$**.**

- $G(0) = 1;\ \ G(n+1) = 2 \cdot G(n)$ **for all** $n \geq\ 0$**.**

- $0! = 1;\ \ (n+1)! = (n+1) \cdot n!$ **for all** $n \geq\ 0$**.**

- $H(0) = 1;\ \ H(n+1) = 2^{H(n)}$ **for all** $n \geq\ 0$**.**

More Recursive Definitions

Suppose that $h: \mathbb{N} \to \mathbb{R}$.

Then we have familiar summation notation:

$\sum_{i=0}^{0} h(i) = h(0)$

$\sum_{i=0}^{n+1} h(i) = h(n+1) + \sum_{i=0}^{n} h(i)$ **for** $n \geq 0$

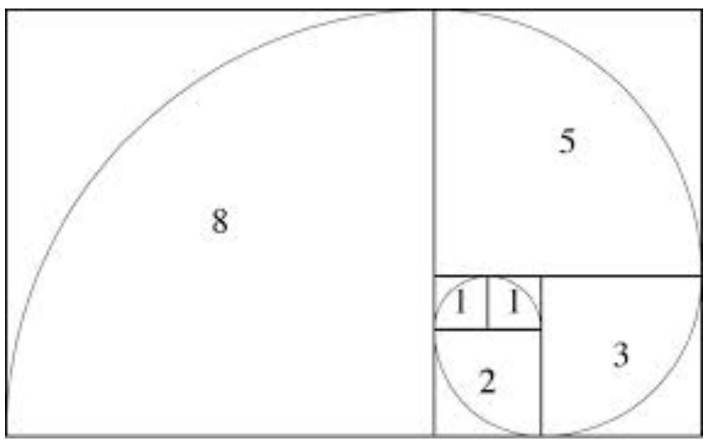
There is also product notation:

$\prod_{i=0}^{0} h(i) = h(0)$

$\prod_{i=0}^{n+1} h(i) = h(n+1) \cdot \prod_{i=0}^{n} h(i)$ **for** $n \geq 0$

Fibonacci Numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$



Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ *is true for every integer j from* b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be** "$f_n < 2^n$". **We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be** "$f_n < 2^n$". **We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0=0 < 1= 2^0$ **so** P(0) **is true.**

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

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2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$, **we have** $f_j < 2^j$ **for every integer** j **from** 0 **to** k.

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

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2. **Base Case:** $f_0=0 < 1= 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$, **we have** $f_j < 2^j$ **for every integer** j **from** 0 **to** k.
4. **Inductive Step:** **Goal: Show** P(k+1); **that is,** $f_{k+1} < 2^{k+1}$

$\boldsymbol{f_0 = 0} \quad \boldsymbol{f_1 = 1}$
$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$ for all $\boldsymbol{n \geq 2}$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$**, we have** $f_j < 2^j$ **for every integer** j **from** 0 **to** k**.**
4. **Inductive Step:** **Goal: Show** P(k+1)**; that is,** $f_{k+1} < 2^{k+1}$

   Case k+1 = 1:

   Case k+1 ≥ 2:

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be** "$f_n < 2^n$". **We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$, **we have** $f_j < 2^j$ **for every integer** j **from** 0 **to** k.
4. **Inductive Step:** **Goal: Show** P(k+1); **that is,** $f_{k+1} < 2^{k+1}$

   Case $k+1 = 1$: **Then** $f_1 = 1 < 2 = 2^1$ **so** P(k+1) **is true here.**

   Case $k+1 \geq 2$:

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be** "$f_n < 2^n$". **We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0=0 < 1= 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$, **we have** $f_j < 2^j$ **for every integer** j **from** 0 **to** k.
4. **Inductive Step:** **Goal: Show** P(k+1); **that is,** $f_{k+1} < 2^{k+1}$

   <u>Case k+1 = 1</u>: **Then** $f_1 = 1 < 2 = 2^1$ **so** P(k+1) **is true here.**

   <u>Case k+1 ≥ 2</u>: **Then** $f_{k+1} = f_k + f_{k-1}$ **by definition**

   $< 2^k + 2^{k-1}$ **by the IH since** $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) **is true in this case.**

   These are the only cases so P(k+1) **follows.**

$$f_0 = 0 \quad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$**, we have** $f_j < 2^j$ **for every integer** j **from** 0 **to** k.
4. **Inductive Step:** **Goal: Show** P(k+1)**; that is,** $f_{k+1} < 2^{k+1}$

   <u>Case k+1 = 1</u>: **Then** $f_1 = 1 < 2 = 2^1$ **so** P(k+1) **is true here.**

   <u>Case k+1 ≥ 2</u>: **Then** $f_{k+1} = f_k + f_{k-1}$ **by definition**

   $< 2^k + 2^{k-1}$ **by the IH since** $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) **is true in this case.**

   These are the only cases so P(k+1) **follows.**

5. **Therefore by strong induction,**

   $f_n < 2^n$ **for all integers** $n \geq 0$.

$$f_0 = 0 \qquad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

$\boldsymbol{f_0 = 0} \quad \boldsymbol{f_1 = 1}$
$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$ for all $\boldsymbol{n \geq 2}$

Bounding Fibonacci II: $f_n \geq 2^{n/2\,-\,1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2\ -1}$**". We prove that** P(n) **is true for all integers** n ≥ 2 **by strong induction.**

$f_0 = 0 \qquad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**".** **We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**

$\boldsymbol{f_0 = 0} \quad \boldsymbol{f_1 = 1}$
$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$ **for all** $\boldsymbol{n \geq 2}$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be** "$f_n \geq 2^{n/2-1}$". **We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 2$, P(j) **is true for every integer** j **from** 2 **to** k.

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. **Let** P(n) **be** "$f_n \geq 2^{n/2 - 1}$". **We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2 - 1} = 2^0 = 1$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 2$, P(j) **is true for every integer** j **from** 2 **to** k**.**
4. **Inductive Step:** **Goal: Show** P(k+1); **that is,** $f_{k+1} \geq 2^{(k+1)/2 - 1}$

$$f_0 = 0 \quad f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be** "$f_n \geq 2^{n/2-1}$". **We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**
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4. **Inductive Step:** **Goal: Show** P(k+1); **that is,** $f_{k+1} \geq 2^{(k+1)/2-1}$

No need for cases for the definition here:

$f_{k+1} = f_k + f_{k-1}$ **since** $k+1 \geq 2$

Now just want to apply the IH to get P(k) **and** P(k-1)

Problem: Though we can get P(k) **since** $k \geq 2$,

k-1 **may only be** 1 **so we can't conclude** P(k-1)

Solution: Separate cases for when k-1=1 **(or** k+1=3**).**

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2 -1}$**".   We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 2$, P(j) **is true for every integer** j **from** 2 **to** k**.**
4. **Inductive Step:** **Goal: Show** P(k+1)**; that is,** $f_{k+1} \geq 2^{(k+1)/2 -1}$

   Case <u>k = 2</u>**:**

   Case <u>k ≥ 3</u>:

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

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3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 2$, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** **Goal: Show** P(k+1); **that is,** $f_{k+1} \geq 2^{(k+1)/2-1}$

   <u>Case k = 2</u>: **Then** $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   <u>Case k ≥ 3</u>:

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

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   Case $k = 2$: **Then** $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   Case $k \geq 3$: $f_{k+1} = f_k + f_{k-1}$ **by definition**

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ **by the IH since** $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   So P(k+1) **is true in both cases.**
5. **Therefore by strong induction,** $f_n \geq 2^{n/2-1}$ **for all integers** $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a,b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

An informal way to get the idea: Consider an n **step** gcd **calculation starting with** r_{n+1}=a **and** r_n=b:

$$\begin{aligned} r_{n+1} &= q_n r_n + r_{n-1} \\ r_n &= q_{n-1} r_{n-1} + r_{n-2} \\ &\ldots \\ r_3 &= q_2 r_2 + r_1 \\ r_2 &= q_1 r_1 \end{aligned}$$

For all $k \geq 2$, $r_{k-1} = r_{k+1} \bmod r_k$

Now $r_1 \geq 1$ **and each** q_k **must be** ≥ 1. **If we replace all the** q_K**'s by** 1 **and replace** r_1 **by** 1 **, we can only reduce the** r_k**'s. After that reduction,** $r_k=f_k$ **for every** k.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes** n **steps for** $\gcd(a,b)$ **with** $a \geq b > 0$**. Then,** $a \geq f_{n+1}$**.**

We go by strong induction on n**.**

Let P(n) **be "**gcd(a,b) **with** a ≥ b>0 **takes** n **steps** ⟶ a ≥ f_{n+1}**" for all** n ≥ 1**.**

<u>**Base Case**</u>**:** n=1 **Suppose Euclid's Algorithm with** a ≥ b > 0 **takes** 1 **step.**
By assumption, a ≥ b ≥ 1 = f_2 **so** P(1) **holds.**

<u>**Induction Hypothesis**</u>**: Suppose that for some integer** k ≥ 1, P(j) **is true for all integers** j **s.t.** 1 ≤ j ≤ k

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes** n **steps for** $\gcd(a, b)$ **with** $a \geq b > 0$. **Then,** $a \geq f_{n+1}$.

We go by strong induction on n.

Let P(n) **be** "gcd(a,b) **with** a ≥ b>0 **takes** n **steps** ⟶ a ≥ f_{n+1}" **for all** n ≥ 1.

Base Case: n=1 **Suppose Euclid's Algorithm with** a ≥ b > 0 **takes** 1 **step.**
By assumption, a ≥ b ≥ 1 = f_2 **so** P(1) **holds.**

Induction Hypothesis: **Suppose that for some integer** k ≥ 1, P(j) **is true for all integers** j **s.t.** 1 ≤ j ≤ k

Inductive Step: **We want to show:** **if** gcd(a,b) **with** a ≥ b > 0 **takes** k+1 **steps, then** a ≥ f_{k+2}.

Running time of Euclid's algorithm

<u>Induction Hypothesis</u>: Suppose that for some integer $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>Inductive Step</u>: Goal: if $\gcd(a,b)$ **with** $a \geq b > 0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2.}$

Now if $k+1=2$, **then Euclid's algorithm on** a **and** b **can be written as**

$a = q_2 b + r_1$

$b = q_1 r_1$

and $r_1 > 0$.

Also, since $a \geq b > 0$ **we must have** $q_2 \geq 1$ and $b \geq 1$.

So $a = q_2 b + r_1 \geq b + r_1 \geq 1+1 = 2 = f_3 = f_{k+2}$ as required.

Running time of Euclid's algorithm

Induction Hypothesis: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

Inductive Step: **Goal: if** $\gcd(a,b)$ **with** $a \geq b>0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2}$.

Next suppose that $k+1 \geq 3$ **so for the first 3 steps of Euclid's algorithm on** a **and** b **we have**

$$a = q_{k+1} b + r_k$$
$$b = q_k r_k + r_{k-1}$$
$$r_k = q_{k-1} r_{k-1} + r_{k-2}$$

and there are $k-2$ **more steps after this.**

Running time of Euclid's algorithm

Induction Hypothesis: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

Inductive Step: **Goal: if** $\gcd(a,b)$ **with** $a \geq b > 0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2}$.

Next suppose that $k+1 \geq 3$ **so for the first 3 steps of Euclid's algorithm on** a **and** b **we have**

$$a = q_{k+1} b + r_k$$
$$b = q_k r_k + r_{k-1}$$
$$r_k = q_{k-1} r_{k-1} + r_{k-2}$$

and there are $k-2$ **more steps after this. Note that this means that the** $\gcd(b, r_k)$ **takes** k **steps and** $\gcd(r_k, r_{k-1})$ **takes** $k-1$ **steps.**

So since $k, k-1 \geq 1$ **by the IH we have** $b \geq f_{k+1}$ **and** $r_k \geq f_k$.

Running time of Euclid's algorithm

Induction Hypothesis: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

Inductive Step: **Goal: if** $\gcd(a,b)$ **with** $a \geq b > 0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2}$.

Next suppose that $k+1 \geq 3$ **so for the first 3 steps of Euclid's algorithm on** a **and** b **we have**

$$a = q_{k+1} b + r_k$$
$$b = q_k r_k + r_{k-1}$$
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and there are $k-2$ **more steps after this. Note that this means that the** $\gcd(b, r_k)$ **takes** k **steps and** $\gcd(r_k, r_{k-1})$ **takes** $k-1$ **steps.**

So since $k, k-1 \geq 1$ **by the IH we have** $b \geq f_{k+1}$ **and** $r_k \geq f_k$.

Also, since $a \geq b$ **we must have** $q_{k+1} \geq 1$.

So $a = q_{k+1} b + r_k \geq b + r_k \geq f_{k+1} + f_k = f_{k+2}$ **as required.** ■

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Why does this help us bound the running time of Euclid's Algorithm?

We already proved that $f_n \geq 2^{n/2\,-\,1}$ so $f_{n+1} \geq 2^{(n-1)/2}$

Therefore: if Euclid's Algorithm takes n steps
for $\gcd(a, b)$ with $a \geq b > 0$
then $a \geq 2^{(n-1)/2}$

so $(n-1)/2 \leq \log_2 a$ or $n \leq 1 + 2\log_2 a$
i.e., # of steps $\leq$ 1 + twice the # of bits in a.

Recursive Definition of Sets

Recursive Definition

- **Basis Step:** $0 \in S$
- **Recursive Step:** If $x \in S$, then $x + 2 \in S$
- **Exclusion Rule:** Every element in S follows from basis steps and a finite number of recursive steps.

Recursive Definitions of Sets

Natural numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+2 \in S$

Powers of 3:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$.

Basis: $[0, 0] \in S$, $[1, 1] \in S$

Recursive: If $[n-1, x] \in S$ and $[n, y] \in S$, then $[n+1, x + y] \in S$. ?

Recursive Definitions of Sets

Natural numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+1 \in S$

Even numbers

Basis: $0 \in S$

Recursive: If $x \in S$, then $x+2 \in S$

Powers of 3:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$.

Basis: $[0, 0] \in S$, $[1, 1] \in S$

Recursive: If $[n-1, x] \in S$ and $[n, y] \in S$, then $[n+1, x + y] \in S$.

Fibonacci numbers

Recursive Definitions of Sets: General Form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma$ (ε is the empty string)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same backwards and forwards

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$

All Binary Strings with no 1’s before 0’s

All Binary Strings with no 1's before 0's

Basis:

$\varepsilon \in S$

Recursive:

If $x \in S$, **then** $0x \in S$

If $x \in S$, **then** $x1 \in S$

Function Definitions on Recursively Defined Sets

Length:

$len(\varepsilon) = 0$

$len(wa) = 1 + len(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$