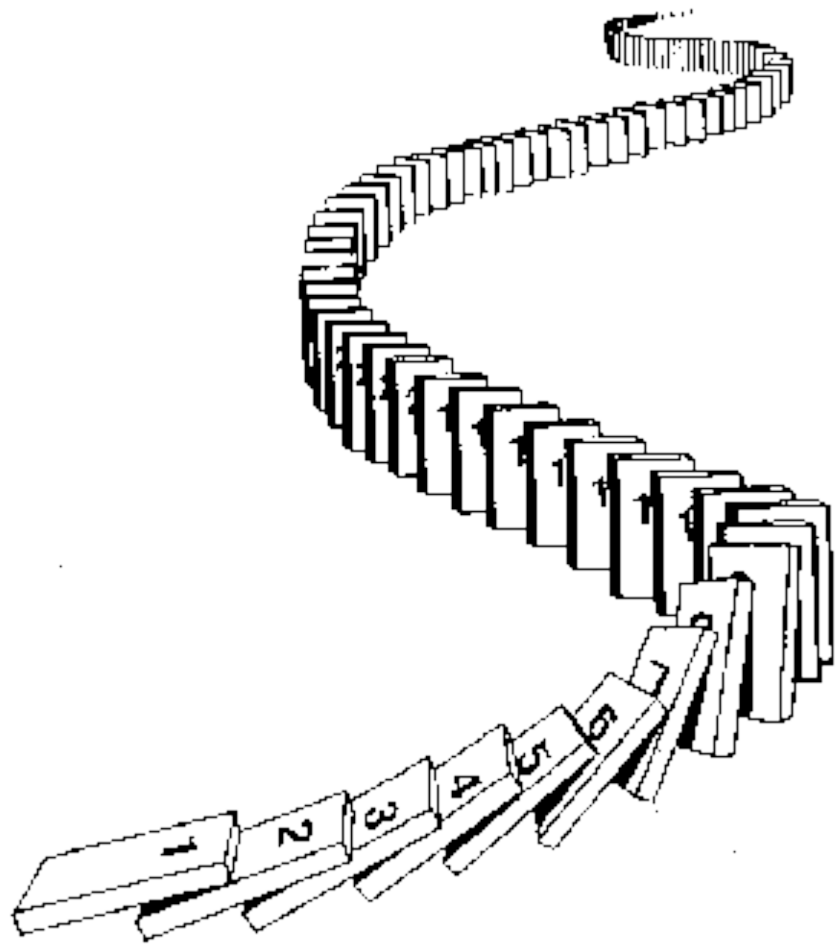
# CSE 311: Foundations of Computing

## Lecture 15: Induction & Strong Induction



# Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq 0$ by induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   *Use the goal to figure out what you need.*

   *Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$ !!)*
5. "Conclusion: $P(n)$ is true for all integers $n \geq 0$"

# Induction: Changing the start line

- What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer $b$?

- Define predicate $Q(k) = P(k + b)$ for all $k$.
  - Then $\forall n\, Q(n) \equiv \forall n \geq b\;\; P(n)$

- Ordinary induction for $Q$:
  - Prove $Q(0) \equiv P(b)$
  - Prove
    $\forall k\, \big(Q(k) \longrightarrow Q(k+1)\big) \equiv \forall k \geq b \big(P(k) \longrightarrow P(k+1)\big)$

# Inductive Proofs In 5 Easy Steps

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2. "Base Case:" Prove $P(\boldsymbol{b})$
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   Assume $P(k)$ is true for some arbitrary integer $k \geq \boldsymbol{b}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

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## Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

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4. **Inductive Step:**

**Goal: Show** P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3$

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$3^{k+1} = 3(3^k)$

$\geq 3(k^2+3)$ **by the IH**

$= k^2+2k^2+9$

$\geq k^2+2k+4 = (k+1)^2+3$ **since** $k \geq 1$.

**Therefore** P(k+1) **is true.**

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   $$
   \begin{aligned}
   3^{k+1} &= 3(3^k) \\
   &\geq 3(k^2+3) \text{ by the IH} \\
   &= k^2+2k^2+9 \\
   &\geq k^2+2k+4 = (k+1)^2+3 \text{ since } k \geq 1.
   \end{aligned}
   $$

   **Therefore** P(k+1) **is true.**

5. **Thus** P(n) **is true for all integers** $n \geq 2$**, by induction.**

# Recall: Induction Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

**How do the givens prove P(5)?**

P(0)→P(1) P(1)→P(2) P(2)→P(3) P(3)→P(4 ) P(4)→P(5)

$P(0)$ $P(1)$ $P(2)$ $P(3)$ $P(4)$ $P(5)$

# Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

**How do the givens prove P(5)?**

P(0)→P(1) P(1)→P(2) P(2)→P(3) P(3)→P(4 ) P(4)→P(5)

$P(0)$ $P(1)$ $P(2)$ $P(3)$ $P(4)$ $P(5)$

**We made it harder than we needed to …**

**When we proved $P(2)$ we knew BOTH $P(0)$ and $P(1)$**

**When we proved $P(3)$ we knew $P(0)$ and $P(1)$ and $P(2)$**

**When we proved $P(4)$ we knew $P(0)$, $P(1)$, $P(2)$, $P(3)$**

**etc.**

**That's the essence of the idea of Strong Induction.**

# Strong Induction

$$P(0)$$

$$\forall k\ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \rightarrow P(k+1)\Big)$$

---

$$\therefore \forall n\ P(n)$$

# Strong Induction

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$$\therefore \forall n \, P(n)$$

**Strong induction for $P$ follows from ordinary induction for $Q$ where**

$$Q(k) = P(0) \land P(1) \land P(2) \land \cdots \land P(k)$$

**Note that $Q(0) \equiv P(0)$ and $Q(k+1) \equiv Q(k) \;\land P(k+1)$ and $\forall n \, Q(n) \equiv \forall n \, P(n)$**

# Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(k)$ is true"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   *Use the goal to figure out what you need.*

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5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

# *Strong* Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ *is true for every integer j from* $b$ to $k$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   *Use the goal to figure out what you need.*

   *Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.*
   *(Don't assume $P(k+1)$ !!)*
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

# Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

48 = 2 • 2 • 2 • 2 • 3
591 = 3 • 197
45,523 = 45,523
321,950 = 2 • 5 • 5 • 47 • 137
1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

**We use strong induction to prove that a factorization into primes exists, but not that it is unique.**

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# Every integer $\geq 2$ is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n $\geq$ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   **Therefore** P(2) **is true.**

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   **Case:** k+1 **is prime: Then by definition** k+1 **is a product of primes**
   **Case:** k+1 **is composite: Then** k+1=ab **for some integers** a **and** b
   **where** 2 ≤ a, b ≤ k.

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2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
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   <u>**Case:** k+1 **is prime**</u>: **Then by definition** k+1 **is a product of primes**
   <u>**Case:** k+1 **is composite:**</u> **Then** k+1=ab **for some integers** a **and** b
   **where** $2 \leq a, b \leq k$. **By our IH**, P(a) **and** P(b) **are true so we have**
   $a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$
   **for some primes** $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.
   **Thus,** $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**
   **Since** $k \geq 1$, **one of these cases must happen and so** P(k+1) **is true.**

# Every integer $\geq 2$ is a product of primes.

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2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   **Therefore** P(2) **is true.**
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   **<u>Case: k+1 is prime</u>: Then by definition** k+1 **is a product of primes**

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   **for some primes** $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

   **Thus,** $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

   **Since** k ≥ 2, **one of these cases must happen and so** P(k+1) **is true.**
5. **Thus** P(n) **is true for all integers** $n \geq 2$, **by strong induction.**

# Strong Induction is particularly useful when…

…we need to analyze methods that on input $k$ make a recursive call for an input different from $k - 1$.

e.g.: Recursive Modular Exponentiation:

– For exponent $k > 0$ it made a recursive call with exponent $\mathrm{j} = k/2$ when $k$ was even or $\mathrm{j} = k - 1$ when $k$ was odd.

We won't analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.

# Recursive definitions of functions

- $F(0) = 0;\ \ F(n+1) = F(n) + 1$ **for all** $n \geq\ 0$**.**

- $G(0) = 1;\ \ G(n+1) = 2 \cdot G(n)$ **for all** $n \geq\ 0$**.**

- $0! = 1;\ \ (n+1)! = (n+1) \cdot n!$ **for all** $n \geq\ 0$**.**

- $H(0) = 1;\ \ H(n+1) = 2^{H(n)}$ **for all** $n \geq\ 0$**.**

# Prove $n! \leq n^n$ for all $n \geq 1$

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2. **Base Case** (n=1): $1! = 1 \cdot 0! = 1 \cdot 1 = 1 = 1^1$ **so** P(1) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 1$. **I.e., suppose** $k! \leq k^k$.
4. **Inductive Step:**

   **Goal: Show** P(k+1), **i.e. show** $(k+1)! \leq (k+1)^{k+1}$

   $(k+1)! = (k+1) \cdot k!$ **by definition of !**

   $\leq (k+1) \cdot k^k$ **by the IH and** $k+1 > 0$

   $\leq (k+1) \cdot (k+1)^k$ **since** $k \geq 0$

   $= (k+1)^{k+1}$

   **Therefore** P(k+1) **is true.**
5. **Thus** P(n) **is true for all** $n \geq 1$, **by induction.**