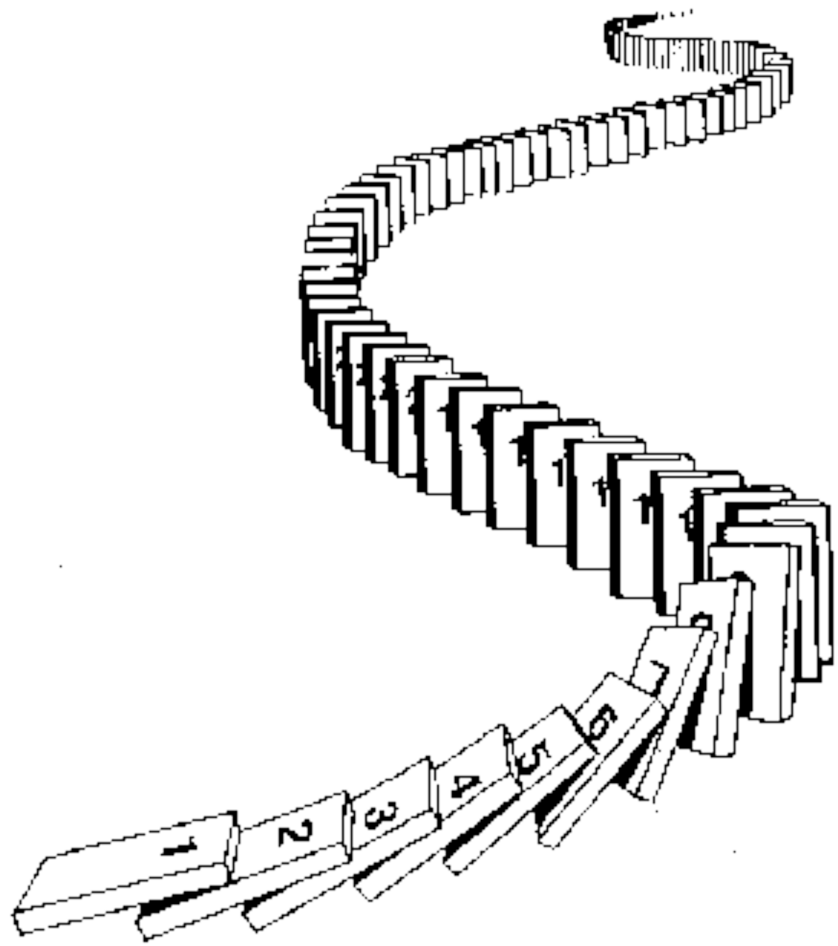
CSE 311: Foundations of Computing

Lecture 14: Induction



Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- **Particularly useful for reasoning about programs!**

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

```
public int f(int x) {
    if (x == 0) { return 0; }
    else { return f(x – 1) + 1; }
}
```

  - f(x) = x for all values of $x \geq 0$ naturally shown by induction.

Prove $\forall a, b, m > 0\ \forall k \in \mathbb{N}\ (a \equiv b \pmod{m} \rightarrow a^k \equiv b^k \pmod{m})$

Let $a, b, m > 0 \in \mathbb{Z}$ **be arbitrary. Let** $k \in \mathbb{N}$ **be arbitrary. Suppose that** $a \equiv b \pmod{m}$**.**

We know $\big(a \equiv b \pmod{m} \land a \equiv b \pmod{m}\big) \rightarrow a^2 \equiv b^2 \pmod{m}$ **by multiplying congruences. So, applying this repeatedly, we have:**

$$\big(a \equiv b \pmod{m} \land a \equiv b \pmod{m}\big) \rightarrow a^2 \equiv b^2 \pmod{m}$$
$$\big(a^2 \equiv b^2 \pmod{m} \land a \equiv b \pmod{m}\big) \rightarrow a^3 \equiv b^3 \pmod{m}$$

…

$$\Big(a^{k-1} \equiv b^{k-1} \pmod{m} \land a \equiv b \pmod{m}\Big) \rightarrow a^k \equiv b^k \pmod{m}$$

The "…"s is a problem! We don't have a proof rule that allows us to say "do this over and over".

But there such a property of the natural numbers!

Domain: Natural Numbers

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

How do the givens prove P(5)?

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

P(0)→P(1)   P(1)→P(2)   P(2)→P(3)   P(3)→P(4)   P(4)→P(5)

$P(0)$ $P(1)$ $P(2)$ $P(3)$ $P(4)$ $P(5)$

First, we have P(0).
Since P(n) ⟶ P(n+1) for all n, we have P(0) ⟶ P(1).
Since P(0) is true and P(0) ⟶ P(1), by Modus Ponens, P(1) is true.
Since P(n) ⟶ P(n+1) for all n, we have P(1) ⟶ P(2).
Since P(1) is true and P(1) ⟶ P(2), by Modus Ponens, P(2) is true.

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)

4. $\forall$k (P(k) $\rightarrow$ P(k+1))
5. $\forall$n P(n) Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0

3. $P(k) \rightarrow P(k+1)$
4. $\forall k\ (P(k) \rightarrow P(k+1))$ Intro $\forall$: 2, 3
5. $\forall n\ P(n)$ Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{P(0) \qquad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n\; P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
   - 3.1. Assume that P(k) is true
   - 3.2. ...
   - 3.3. Prove P(k+1) is true
3. $P(k) \rightarrow P(k+1)$ — Direct Proof Rule
4. $\forall k\,(P(k) \rightarrow P(k+1))$ — Intro $\forall$: 2, 3
5. $\forall n\, P(n)$ — Induction: 1, 4

Translating to an English Proof

$$\frac{P(0) \qquad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n\;\; P(n)}$$

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0
    3.1. Assume that P(k) is true — **Inductive Hypothesis**

    3.2. ...
    3.3. Prove P(k+1) is true — **Inductive Step**

3.	$P(k) \rightarrow P(k+1)$	Direct Proof Rule
4.	$\forall k\,(P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5.	$\forall n\, P(n)$	Induction: 1, 4

Conclusion

Translating To An English Proof

1. Prove P(0) — **Base Case**
2. Let k be an arbitrary integer ≥ 0 — **Inductive Hypothesis**
    3.1. Assume that P(k) is true
    3.2. ... — **Inductive Step**
    3.3. Prove P(k+1) is true

3.	$P(k) \rightarrow P(k+1)$	Direct Proof Rule
4.	$\forall k\, (P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5.	$\forall n\, P(n)$	Induction: 1, 4

Conclusion

Induction Proof Template

[...Define P(n)...]

We will show that $P(n)$ **is true for every** $n \in \mathbb{N}$ **by Induction.**

Base Case: *[...proof of* $P(0)$ *here...]*

Induction Hypothesis:

Suppose that $P(k)$ **is true for some** $k \in \mathbb{N}$**.**

Induction Step:

We want to prove that $P(k+1)$ **is true.**

[...proof of $P(k+1)$ *here...]*

The proof of $P(k+1)$ ***must*** *invoke the IH somewhere.*

So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Proof:

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.

Good that we have induction!

Prove $1 + 2 + 4 + \ldots + 2^{n} = 2^{n+1} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) i**s true for all natural numbers by induction.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0.

   I.e, suppose $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$.

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0.
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

   $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ **by IH**

   Adding 2^{k+1} **to both sides, we get:**

   $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** P(k+1).

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

We can calculate

$$\begin{aligned} 1 + 2 + \ldots + 2^k + 2^{k+1} &= (1+2+\ldots+2^k) + 2^{k+1} \\ &= 2^{k+1} - 1 + 2^{k+1} && \textbf{by the IH} \\ &= 2(2^{k+1}) - 1 \\ &= 2^{k+2} - 1, \end{aligned}$$

which is exactly P(k+1).

Alternative way of writing the inductive step

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

   We can calculate

$$
\begin{aligned}
1 + 2 + \ldots + 2^k + 2^{k+1} &= (1+2+ \ldots + 2^k) + 2^{k+1} \\
&= 2^{k+1} - 1 + 2^{k+1} && \textbf{by the IH} \\
&= 2(2^{k+1}) - 1 \\
&= 2^{k+2} - 1,
\end{aligned}
$$

   which is exactly P(k+1).

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

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2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0.
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** 1 + 2 + …+ k+ (k+1) = (k+1)(k+2)/2

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**.**
4. **Induction Step:**

$$
\begin{aligned}
1 + 2 + \ldots + k + (k+1) &= (1 + 2 + \ldots + k) + (k+1) \\
&= k(k+1)/2 + (k+1) \quad \textbf{by IH} \\
&= (k+1)(k/2 + 1) \\
&= (k+1)(k+2)/2
\end{aligned}
$$

   So, we have shown 1 + 2 + … + k + (k+1) = (k+1)(k+2)/2**, which is exactly** P(k+1)**.**

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

Another example of a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- …

Prove: $3 \mid (2^{2n}-1)$ **for all** $n \geq 0$

Prove: $3 \mid (2^{2n} - 1)$ for all $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) i**s true for all natural numbers by induction.**
2. **Base Case** (n=0):

Prove: $3 \mid (2^{2n}-1)$ for all $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}-1=1-1=0=3\cdot 0$ **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** $3 \mid (2^{2(k+1)} - 1)$

Prove: $3 \mid (2^{2n} - 1)$ for all $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}-1=1-1=0=3\cdot 0$ **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**.**
4. **Induction Step:**

   By IH, $3 \mid (2^{2k} - 1)$ **so** $2^{2k} - 1 = 3j$ **for some integer** j

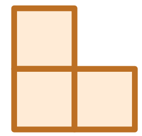
   So $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 4(2^{2k}) - 1 = 4(3j+1) - 1$

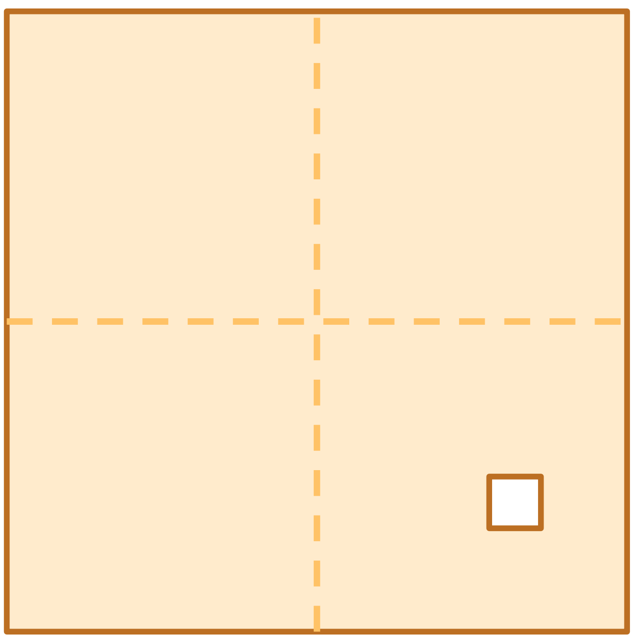
   $= 12j+3 = 3(4j+1)$

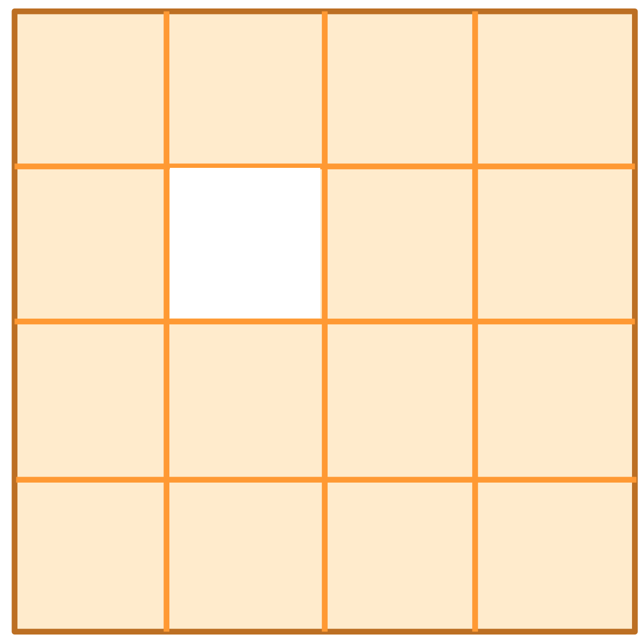
   Therefore $3 \mid (2^{2(k+1)} - 1)$ **which is exactly** P(k+1)**.**

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

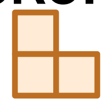
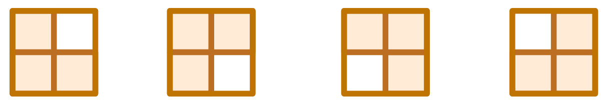
Checkerboard Tiling

- **Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with:** 





Checkerboard Tiling

1. **Let** P(n) **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove P(n) **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** n=1 

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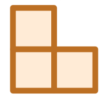
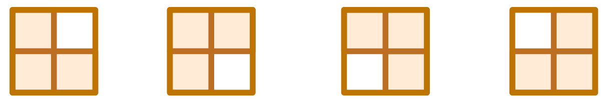
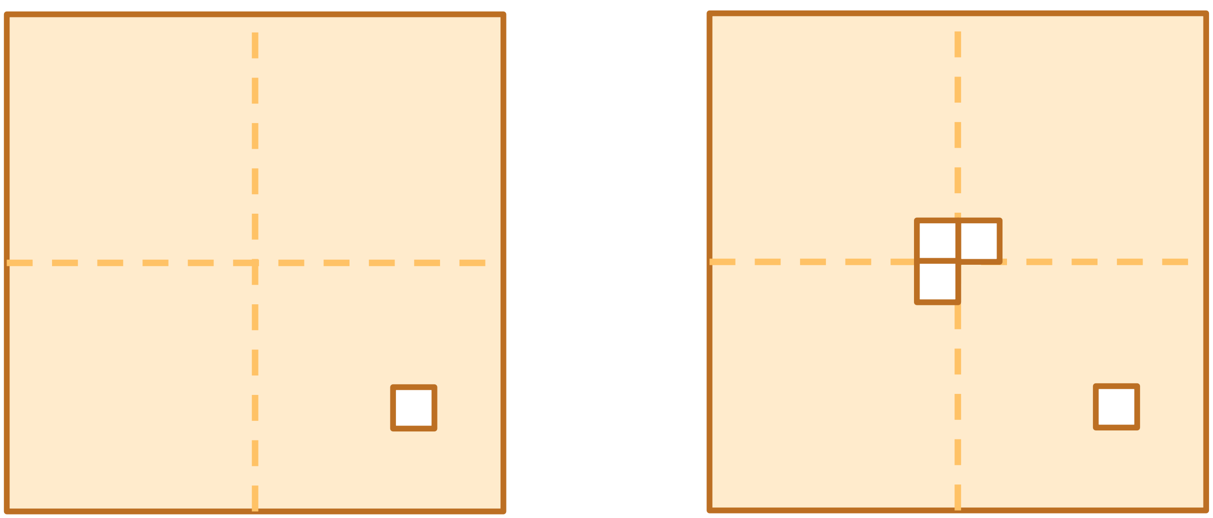
1. Let P(n) be any $2^n \times 2^n$ checkerboard with one square removed can be tiled with  .
   We prove P(n) for all $n \geq 1$ by induction on n.
2. Base Case: n=1 
3. Inductive Hypothesis: Assume P(k) for some arbitrary integer $k \geq 1$
4. Inductive Step: Prove P(k+1)



Apply IH to each quadrant then fill with extra tile.