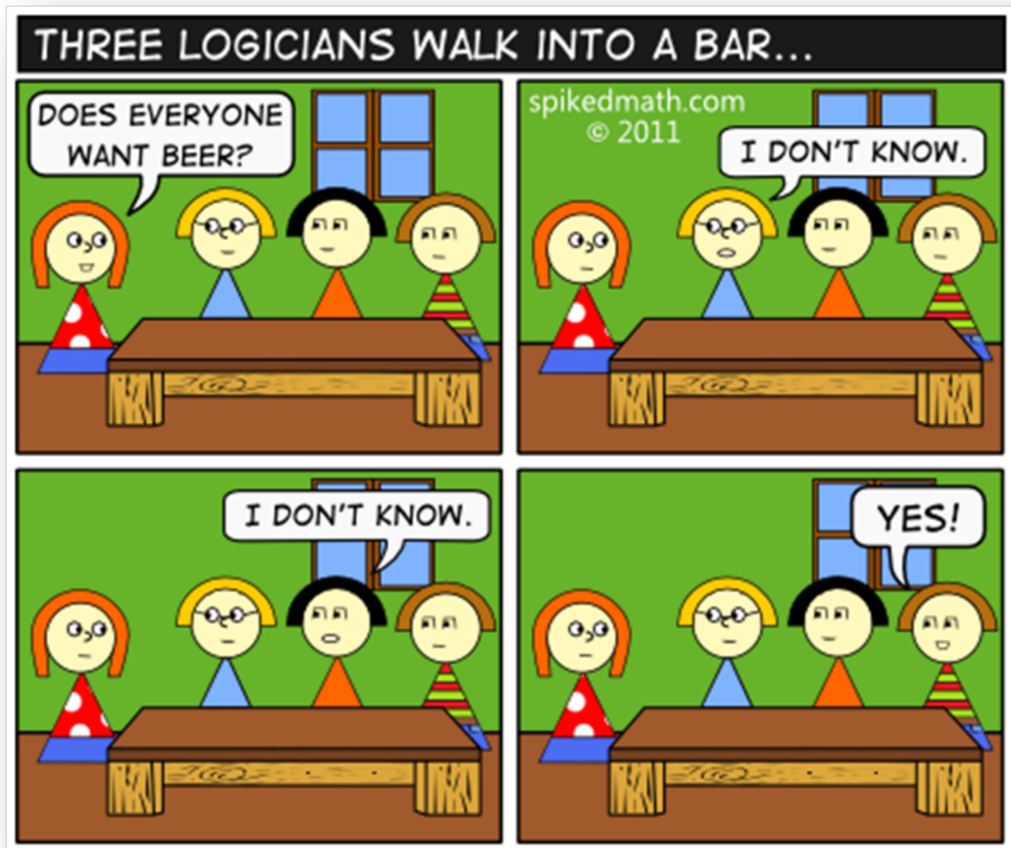
CSE 311: Foundations of Computing

Lecture 6: More Predicate Logic



Last class

Canonical Forms

– sum-of-products and product-of-sums

– both are useful

Corollaries of construction:

– any function can be formed with just $\neg$, $\vee$, $\wedge$

– actually, just $\neg$, $\vee$ (De Morgan's laws)

– actually, just A (HW1 Q3 (a-c))

NAND and NOR also have this property

Last class: Predicates

Predicate

– **A function that returns a truth value, e.g.,**

Cat(x) ::= "x is a cat"

Prime(x) ::= "x is prime"

HasTaken(x, y) ::= "student x has taken course y"

LessThan(x, y) ::= "x < y"

Sum(x, y, z) ::= "x + y = z"

GreaterThan5(x) ::= "x > 5"

HasNChars(s, n) ::= "string s has length n"

Predicates can have varying numbers of arguments and input types.

Last class: Domain of Discourse

For ease of use, we define one “type”/“domain” that we work over. This non-empty set of objects is called the “**domain of discourse**”.

For each of the following, what might the domain be?

(1) “x is a cat”, “x barks”, “x ruined my couch”

(2) “x is prime”, “x = 0”, “x < 0”, “x is a power of two”

(3) “x is a pre-req for z”

Domain of Discourse

For ease of use, we define one "type"/"domain" that we work over. This set of objects is called the "**domain of discourse**".

For each of the following, what might the domain be?

(1) "x is a cat", "x barks", "x ruined my couch"

"mammals" or "sentient beings" or "cats and dogs" or ...

(2) "x is prime", "$x = 0$", "$x < 0$", "x is a power of two"

"numbers" or "integers" or "integers greater than 5" or ...

(3) "x is a pre-req for z"

"courses"

Last Class: Quantifiers

We use *quantifiers* to talk about collections of objects.

$\forall x\ P(x)$

P(x) **is true for every** x **in the domain**

read as "for all x, P of x"

$\exists x\ P(x)$

There is an x **in the domain for which** P(x) **is true**

read as "there exists x, P of x"

Last class: Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Determine the truth values of each of these statements:

$\exists x$ Even(x)

$\forall x$ Odd(x)

$\forall x$ (Even(x) $\vee$ Odd(x))

$\exists x$ (Even(x) $\wedge$ Odd(x))

$\forall x$ Greater(x+1, x)

$\exists x$ (Even(x) $\wedge$ Prime(x))

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Determine the truth values of each of these statements:

$\exists x\ \text{Even}(x)$	**T**	e.g. 2, 4, 6, ...
$\forall x\ \text{Odd}(x)$	**F**	e.g. 2, 4, 6, ...
$\forall x\ (\text{Even}(x) \vee \text{Odd}(x))$	**T**	every integer is either even or odd
$\exists x\ (\text{Even}(x) \wedge \text{Odd}(x))$	**F**	no integer is both even and odd
$\forall x\ \text{Greater}(x+1, x)$	**T**	adding 1 makes a bigger number
$\exists x\ (\text{Even}(x) \wedge \text{Prime}(x))$	**T**	Even(2) is true and Prime(2) is true

Statements with Quantifiers

Domain of Discourse
Positive Integers

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Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\ \exists y\ \text{Greater}(y, x)$

$\forall x\ \exists y\ \text{Greater}(x, y)$

$\forall x\ \exists y\ (\text{Greater}(y, x) \wedge \text{Prime}(y))$

$\forall x\ (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

$\exists x\ \exists y\ (\text{Sum}(x, 2, y) \wedge \text{Prime}(x) \wedge \text{Prime}(y))$

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\ \exists y\ \text{Greater}(y, x)$

For every positive integer x, there is a positive integer y, such that y > x.

$\exists y\ \forall x\ \text{Greater}(y, x)$

There is a positive integer y such that, for every pos. int. x, we have y > x.

$\forall x\ \exists y\ (\text{Greater}(y, x) \wedge \text{Prime}(y))$

For every positive integer x, there is a pos. int. y such that y > x and y is prime.

$\forall x\ (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

For each positive integer x, if x is prime, then x = 2 or x is odd.

$\exists x\ \exists y\ (\text{Sum}(x, 2, y) \wedge \text{Prime}(x) \wedge \text{Prime}(y))$

There exist positive integers x and y such that x + 2 = y and x and y are prime.

Statements with Quantifiers (Natural Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\ \exists y\ \text{Greater}(y, x)$

For every positive integer, there is a larger positive integer.

$\exists y\ \forall x\ \text{Greater}(y, x)$

There is a largest positive integer.

$\forall x\ \exists y\ (\text{Greater}(y, x) \wedge \text{Prime}(y))$

For every positive integer, there is a larger number that is prime.

$\forall x\ (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

Every prime number is either 2 or odd.

$\exists x\ \exists y\ (\text{Sum}(x, 2, y) \wedge \text{Prime}(x) \wedge \text{Prime}(y))$

There exist prime numbers that differ by two.

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

"Red cats like tofu"

"Some red cats don't like tofu"

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

"Red cats like tofu"

$$\forall x\ ((\mathrm{Red}(x) \land \mathrm{Cat}(x)) \to \mathrm{LikesTofu}(x))$$

"Some red cats don't like tofu"

$$\exists y\ ((\mathrm{Red}(y) \land \mathrm{Cat}(y)) \land \neg \mathrm{LikesTofu}(y))$$

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

When putting two predicates together like this, we use an "and".

"Red cats like tofu"

When restricting to a smaller domain in a "for all" we use **implication.**

When there's no leading quantification, it means "for all".

"Some red cats don't like tofu"

When restricting to a smaller domain in an "exists" we use **and.**

"Some" means "there exists".

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= "x is a purple fruit"

(*) $\forall$x PurpleFruit(x) ("All fruits are purple")

What is the negation of (*)?

(a) "there exists a purple fruit"

(b) "there exists a non-purple fruit"

(c) "all fruits are not purple"

Try your intuition! Which one "feels" right?

Key Idea: In every domain, exactly one of a statement and its negation should be true.

Negations of Quantifiers

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Key Idea: In every domain, exactly one of a statement and its negation should be true.

Domain of Discourse
{plum}

(*), (a)

Domain of Discourse
{apple}

(b), (c)

Domain of Discourse
{plum, apple}

(a), (b)

Negations of Quantifiers

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(*) $\forall x$ PurpleFruit(x) ("All fruits are purple")

What is the negation of (*)?

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Key Idea: In every domain, exactly one of a statement and its negation should be true.

Domain of Discourse
{plum}

(*), (a)

Domain of Discourse
{apple}

(b), (c)

Domain of Discourse
{plum, apple}

(a), (b)

The only choice that ensures exactly one of the statement and its negation is (b).

De Morgan's Laws for Quantifiers

$$\boxed{\begin{array}{c} \neg\forall x\, P(x) \equiv \exists x\, \neg P(x) \\ \neg\exists x\, P(x) \equiv \forall x\, \neg P(x) \end{array}}$$

De Morgan's Laws for Quantifiers

$$\neg\forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg\exists x\, P(x) \equiv \forall x \neg P(x)$$

"There is no largest integer"

$$\neg \exists x \forall y\ (x \geq y)$$
$$\equiv \forall x \neg \forall y\ (x \geq y)$$
$$\equiv \forall x\ \exists y \neg (x \geq y)$$
$$\equiv \forall x\ \exists y\ (y > x)$$

"For every integer, there is a larger integer"

Scope of Quantifiers

$\exists x \; (P(x) \land Q(x))$ **vs.** $\exists x\, P(x) \land \exists x\, Q(x)$

scope of quantifiers

$\exists x \; (P(x) \wedge Q(x))$ **vs.** $\exists x \; P(x) \wedge \exists x \; Q(x)$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

Scope of Quantifiers

Example: NotLargest(x) $\equiv \exists\, y$ Greater (y, x)
$\equiv \exists\, z$ Greater (z, x)

truth value:

doesn't depend on y **or** z **"bound variables"**

does depend on x **"free variable"**

quantifiers only act on free variables of the formula they quantify

$$\forall\, x\, (\exists\, y\, (P(x,y) \rightarrow \forall\, x\, Q(y, x)))$$

Quantifier "Style"

$$\forall x(\exists y\ (P(x,y) \to \forall\ x\ Q(y, x)))$$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…

Nested Quantifiers

- **Bound variable names don't matter**

$$\forall x\ \exists y\ P(x, y) \equiv \forall a\ \exists b\ P(a, b)$$

- **Positions of quantifiers can sometimes change**

$$\forall x\ (Q(x) \land \exists y\ P(x, y)) \equiv \forall x\ \exists y\ (Q(x) \land P(x, y))$$

- **But: order is important...**

Quantifier Order Can Matter

Domain of Discourse
Integers OR {1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= "$x \geq y$"

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

"There is a number greater than or equal to all numbers."

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)))$$

"Every number has a number greater than or equal to it."

$$\forall y\ \exists x\ \text{GreaterEq}(x, y)))$$

The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Quantification with Two Variables

expression	when **true**	when **false**
$\forall x \forall y\, P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y\, P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y\, P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \forall x\, P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If p and p $\rightarrow$ q are both true then q must be true

- Write this rule as $$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

- Given:
  - If it is Friday, then you have a 311 class today.
  - It is Friday.

- Therefore, by Modus Ponens:
  - You have a 311 class today.

My First Proof!

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.		
5.		

My First Proof!

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.	q	MP: 1, 2
5.	r	MP: 3, 4

Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	Given
2.	$\neg q$	Given
3.	$\neg q \rightarrow \neg p$	Contrapositive: 1
4.	$\neg p$	MP: 2, 3

Inference Rules

- Each inference rule is written as:

  ...which means that if both A and B are true then you can infer C and you can infer D.

  $$\frac{A, B}{\therefore\ C, D}$$

  – For rule to be correct $(A \wedge B) \rightarrow C$ and $(A \wedge B) \rightarrow D$ must be a tautologies

- Sometimes rules don't need anything to start with. These rules are called axioms:
  – e.g. *Excluded Middle Axiom*

  $$\frac{}{\therefore\ p \vee \neg p}$$

Simple Propositional Inference Rules

Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

$$\frac{p \wedge q}{\therefore\ p,\ q} \qquad \frac{p,\ q}{\therefore\ p \wedge q}$$

$$\frac{p \vee q\ ,\ \neg p}{\therefore\ q} \qquad \frac{p}{\therefore\ p \vee q,\ q \vee p}$$

$$\frac{p,\ p \rightarrow q}{\therefore\ q} \qquad \frac{p \Rightarrow q}{\therefore\ p \rightarrow q}$$

Direct Proof Rule

Not like other rules

Proofs

Show that r follows from p, $p \rightarrow q$ and $(p \wedge q) \rightarrow r$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

$$\frac{p \wedge q}{\therefore\ p,\ q}$$

$$\frac{p,\ q}{\therefore\ p \wedge q}$$

Proofs

Show that r follows from $p, p \to q,$ and $p \land q \to r$

1.	p	Given
2.	$p \to q$	Given
3.	q	MP: 1, 2
4.	$p \land q$	Intro $\land$: 1, 3
5.	$p \land q \to r$	Given
6.	r	MP: 4, 5

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!

$$\dfrac{\dfrac{p \qquad \dfrac{p \quad p \to q}{q}\,\text{MP}}{p \land q}\,\text{Intro } \land \qquad p \land q \to r}{r}\,\text{MP}$$

Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \to q$ given

~~2. $(p \vee r) \to q$ intro $\vee$ from 1.~~

Does not follow! e.g . p=**F**, q=**F**, r=**T**