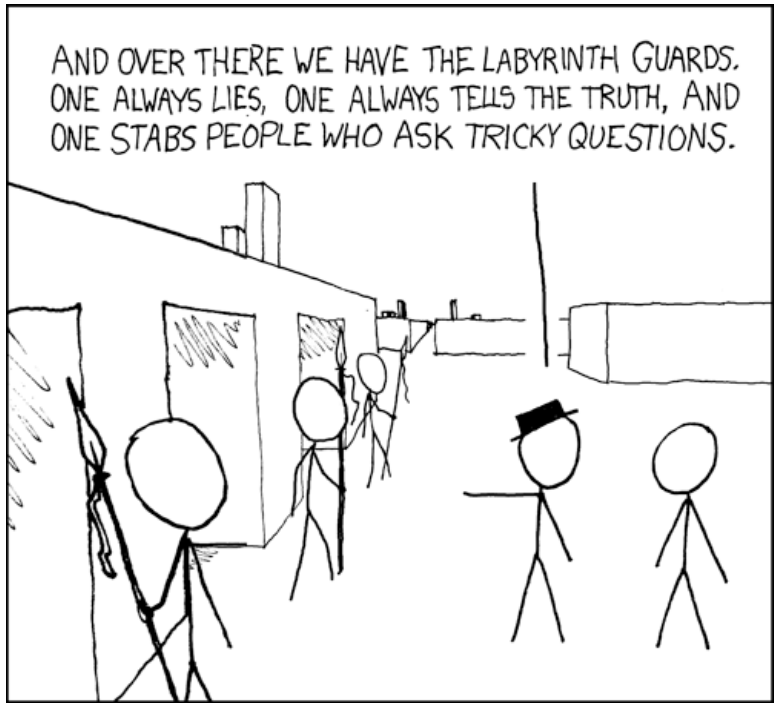
CSE 311: Foundations of Computing

Lecture 3: Digital Circuits & Equivalence



Homework #1

- Posted yesterday. Due Wednesday

- Submit your solution via GradeScope
- You should receive an email invitation <u>tonight</u>
  - If you don't receive one, send e-mail to cse311-staff@cs.washington.edu

- Grading policy on web site updated:
  - solutions must be *legible*
  - submissions requiring work to interpret will lose points
  - useful tips for typesetting solutions on web site

Grading Policies

Our determinations of the correctness of each answer will be made based on the *intention* of the problem, as long as the staff believes that the intention was clear. We will not listen to legalistic arguments about why poor solutions should be considered correct.

We will not debate the amount of points deducted for mistakes. Those are entirely at the descretion of the course staff.

Grading Guidelines

Legibility is critical

You may lose points for solutions that are not legible. Whenever the grader has to spend a noticeable amount of time trying to determine what your submission actually says, they will deduct points.

You can eliminate that possibility by typesetting your solution. (See below for suggestions on how to do so.)

Clarity is important

(more here)

Typesetting Suggestions

LaTeX is the standard tool for typesetting mathematical materials. While it takes some time to learn, it will likely pay for itself in the long run. LaTeX math notation is also supported in some places outside of LaTeX documents, such as on the Piazza message board.

A former CSE 311 instructor, Adam Blank, has provided these documents, which may be useful to you:

- A template LaTeX file to use for CSE 311 homework assigments.
- A tutorial on LaTeX, including specific information on how to use the template.

Note that LaTeX does not need to be installed on your computer. You can use it in a web browser at the Overleaf web site.

While LaTeX is not required for your assignments in CSE 311, it is encouraged. However, LaTeX is not the best tool for every job. In particular, for drawing circuits, finite state machines, and other diagrams, it is often preferable to draw them by hand, take a picture, and include it in your LaTeX document using the `\includegraphics{FILE_NAME}` command.

Last class: Logical Equivalence $A \equiv B$

$A \equiv B$ is an assertion that ***two propositions*** A and B always have the same truth values.

tautology

$A \equiv B$ and $(A \leftrightarrow B) \equiv T$ have the same meaning.

$p \wedge q \equiv q \wedge p$

p	q	$p \wedge q$	$q \wedge p$	$(p \wedge q) \leftrightarrow (q \wedge p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

$p \wedge q \not\equiv q \vee p$

When p=T and q=F, $p \wedge q$ is false, but $q \vee p$ is true

Last class: De Morgan's Laws

De Morgan's Laws

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Last class: Equivalences Related to Implication

Law of Implication

$$p \to q \equiv \neg p \lor q$$

Contrapositive

$$p \to q \equiv \neg q \to \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

Last class: Properties of Logical Connectives

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

One more easy equivalence

Double Negation

$$p \equiv \neg\neg p$$

p	$\neg p$	$\neg\neg p$	$p \leftrightarrow \neg\neg p$
T	F	T	T
F	T	F	T

Last class: Digital Circuits

Computing With Logic

– T corresponds to 1 or "high" voltage

– F corresponds to 0 or "low" voltage

Gates

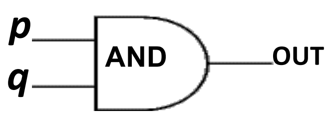
– Take inputs and produce outputs (functions)

– Several kinds of gates

– Correspond to propositional connectives (most of them)

Last class: AND, OR, NOT Gates

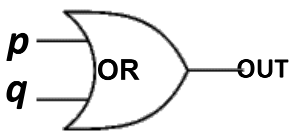
AND Gate



p	q	OUT
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

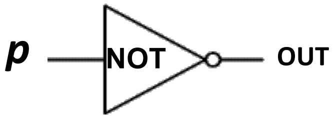
OR Gate



p	q	OUT
1	1	1
1	0	1
0	1	1
0	0	0

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

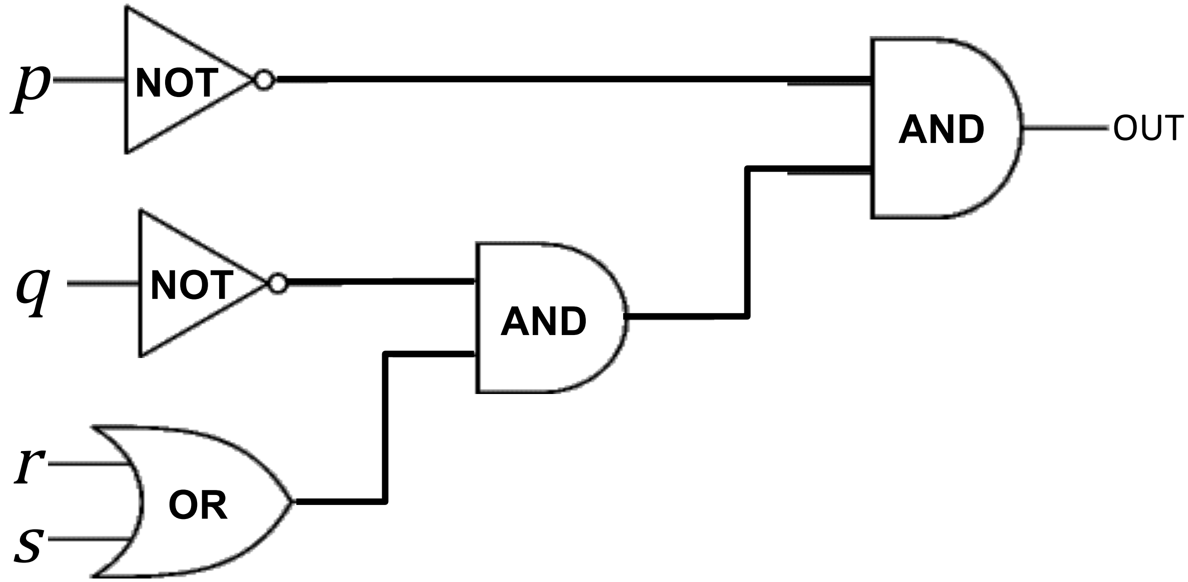
NOT Gate



p	OUT
1	0
0	1

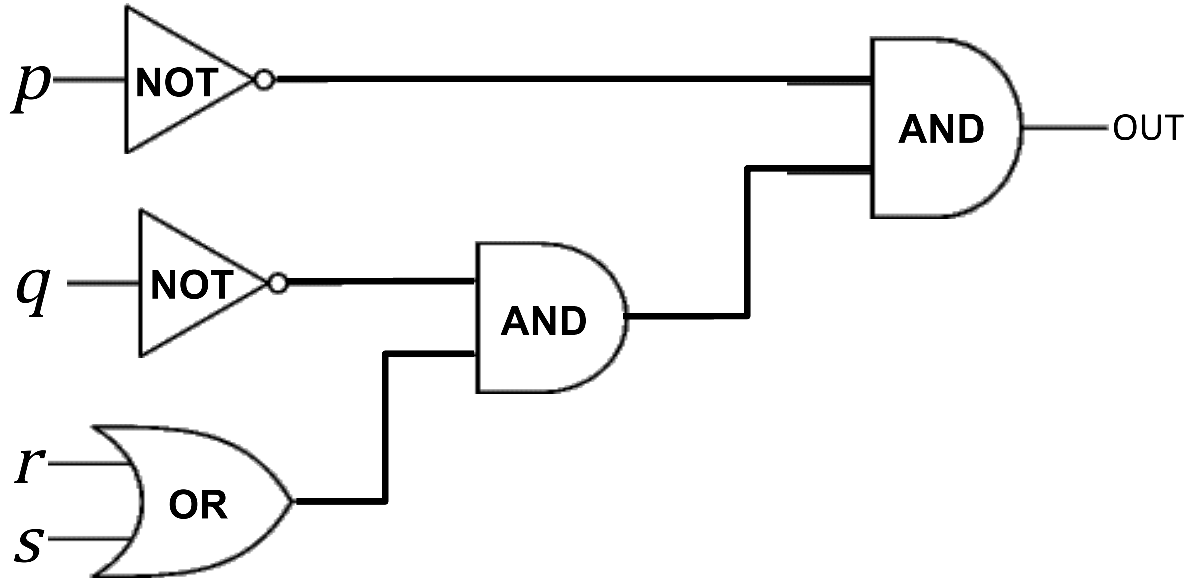
p	$\neg p$
T	F
F	T

Combinational Logic Circuits



Values get sent along wires connecting gates

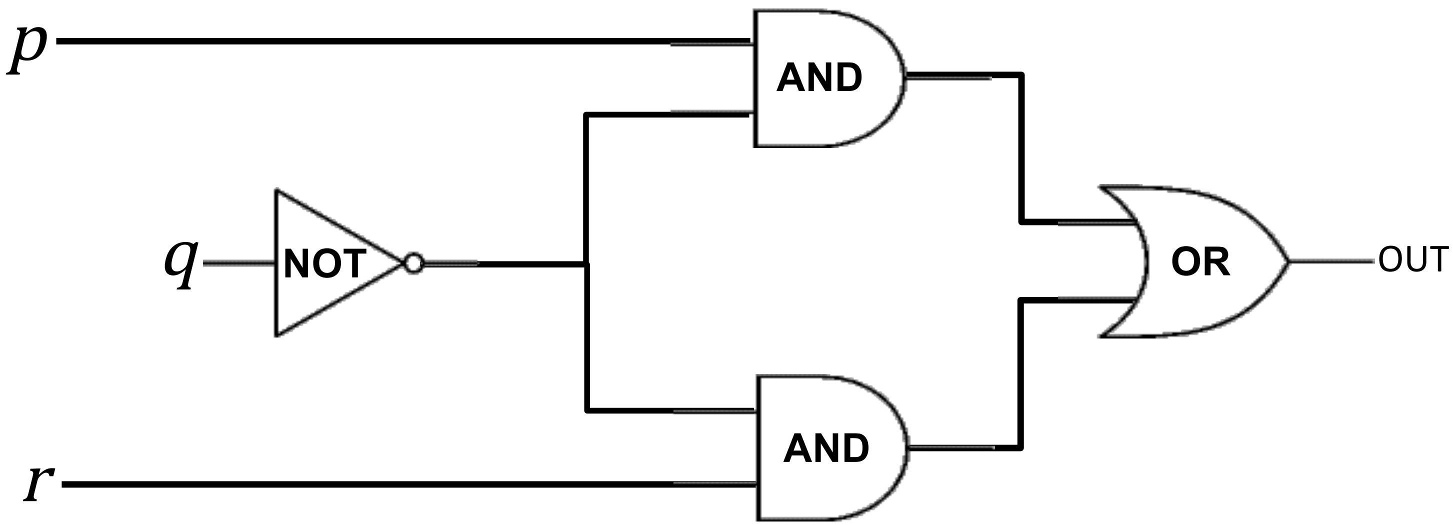
Combinational Logic Circuits



Values get sent along wires connecting gates

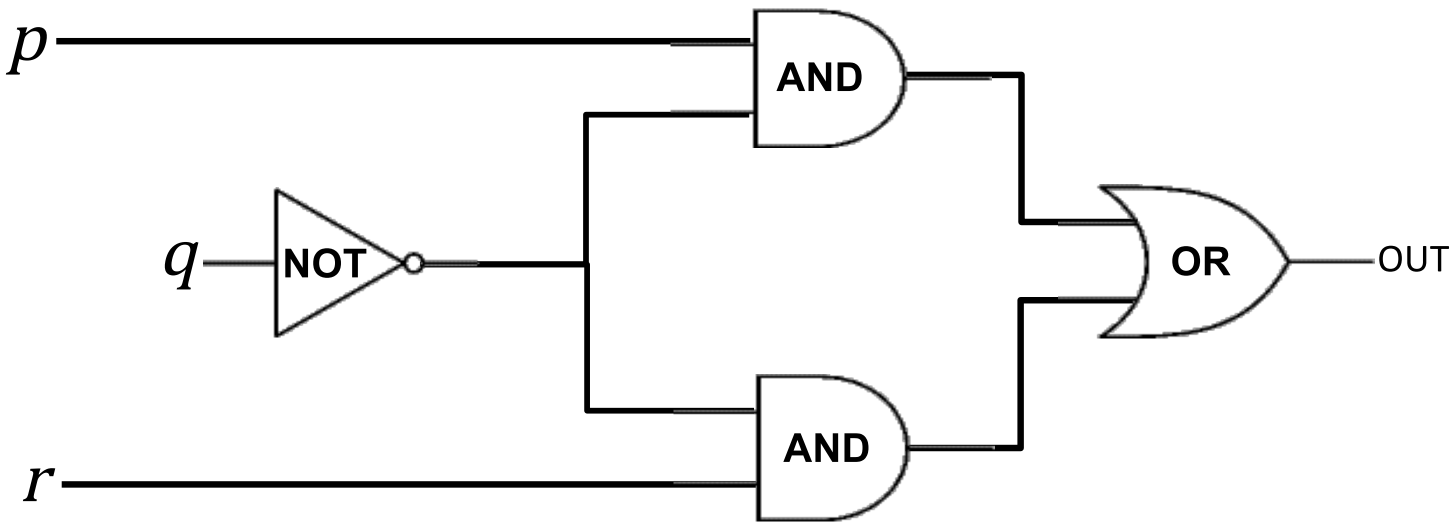
$$\neg p \wedge (\neg q \wedge (r \vee s))$$

Combinational Logic Circuits



Wires can send one value to multiple gates!

Combinational Logic Circuits



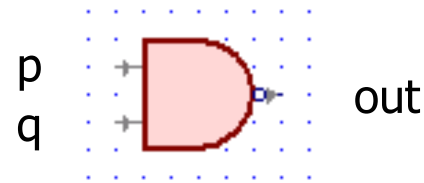
Wires can send one value to multiple gates!

$$(p \wedge \neg q) \vee (\neg q \wedge r)$$

Other Useful Gates

NAND

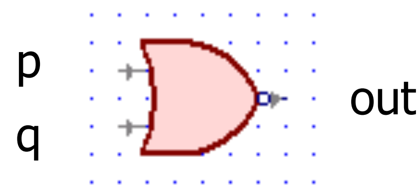
$\neg(p \land q)$



p	q	out
0	0	1
0	1	1
1	0	1
1	1	0

NOR

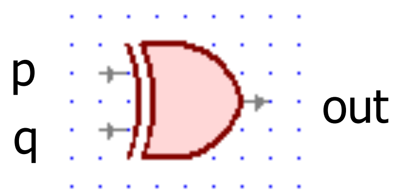
$\neg(p \lor q)$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	0

XOR

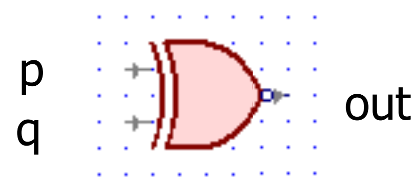
$p \bigoplus q$



p	q	out
0	0	0
0	1	1
1	0	1
1	1	0

XNOR

$p \leftrightarrow q$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	1

Understanding logic and circuits

When do two logic formulas mean the same thing?

When do two circuits compute the same function?

What logical properties can we infer from other ones?

Basic rules of reasoning and logic

- Allow manipulation of logical formulas
  - Simplification
  - Testing for equivalence
- Applications
  - Query optimization
  - Search optimization and caching
  - Artificial Intelligence
  - Program verification

Computing Equivalence

Given two propositions, can we write an algorithm to determine if they are equivalent?

What is the runtime of our algorithm?

Computing Equivalence

Given two propositions, can we write an algorithm to determine if they are equivalent?

Yes! Generate the truth tables for both propositions and check if they are the same for every entry.

What is the runtime of our algorithm?

Every atomic proposition has two possibilities (T, F). If there are n atomic propositions, there are 2^n rows in the truth table.

Another approach: Logical Proofs

To show A is equivalent to B

- Apply a series of logical equivalences to sub-expressions to convert A to B

To show A is a tautology

- Apply a series of logical equivalences to sub-expressions to convert A to **T**

Another approach: Logical Proofs

To show A is equivalent to B

– Apply a series of logical equivalences to sub-expressions to convert A to B

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general proof looks like:

$$p \vee (p \wedge p) \equiv (\qquad\qquad\qquad\qquad)$$
$$\equiv p$$

Another approach: Logical Proofs

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \to q \equiv \neg p \vee q$$

Contrapositive

$$p \to q \equiv \neg q \to \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general proof looks like:

$$p \vee (p \wedge p) \equiv (\qquad\qquad\qquad\qquad)$$
$$\equiv p$$

Logical Proofs

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
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  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
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Biconditional

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Double Negation

$$p \equiv \neg \neg p$$

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general proof looks like:

$$p \vee (p \wedge p) \equiv (\quad p \vee p \quad) \qquad \text{Idempotent}$$
$$\equiv p \qquad \text{Idempotent}$$

Logical Proofs

To show A is a tautology

- Apply a series of logical equivalences to sub-expressions to convert A to **T**

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general proof looks like:

$$
\begin{aligned}
\neg p \vee (p \vee p) &\equiv (\qquad\qquad\qquad) \\
&\equiv (\qquad\qquad\qquad) \\
&\equiv \mathbf{T}
\end{aligned}
$$

Logical Proofs

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
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  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general proof looks like:

$$\neg p \vee (p \vee p) \equiv (\qquad\qquad\qquad)$$
$$\equiv (\qquad\qquad\qquad)$$
$$\equiv \mathrm{T}$$

Logical Proofs

- **Identity**
  - $p \wedge T \equiv p$
  - $p \vee F \equiv p$
- **Domination**
  - $p \vee T \equiv T$
  - $p \wedge F \equiv F$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
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  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$

De Morgan's Laws

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$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

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$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general proof looks like:

$\neg p \vee (p \vee p)$	$\equiv (\neg p \vee p)$	Idempotent
	$\equiv (p \vee \neg p)$	Commutative
	$\equiv T$	Negation

Prove these propositions are equivalent: Option 1

Prove: $p \wedge (p \rightarrow q) \equiv p \wedge q$

Make a Truth Table and show:

$$(p \wedge (p \rightarrow q)) \leftrightarrow (p \wedge q) \equiv \mathbf{T}$$

p	q	$p \rightarrow q$	$(p \wedge (p \rightarrow q))$	$p \wedge q$	$(p \wedge (p \rightarrow q)) \leftrightarrow (p \wedge q)$
T	T	T	T	T	T
T	F	F	F	F	T
F	T	T	F	F	T
F	F	T	F	F	T

Prove these propositions are equivalent: Option 2

Prove: $p \wedge (p \rightarrow q) \equiv p \wedge q$

$$p \wedge (p \rightarrow q) \equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv p \wedge q$$

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
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  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

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$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Prove these propositions are equivalent: Option 2

Prove: $p \wedge (p \rightarrow q) \equiv p \wedge q$

$p \wedge (p \rightarrow q) \equiv p \wedge (\neg p \vee q)$	Law of Implication
$\equiv (p \wedge \neg p) \vee (p \wedge q)$	Distributive
$\equiv \mathbf{F} \vee (p \wedge q)$	Negation
$\equiv (p \wedge q) \vee \mathbf{F}$	Commutative
$\equiv p \wedge q$	Identity

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
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$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg \neg p$$

Prove this is a Tautology: Option 1

$$(p \wedge q) \rightarrow (q \vee p)$$

Make a Truth Table and show:

$$(p \wedge q) \rightarrow (q \vee p) \equiv \mathbf{T}$$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{q \vee p}$	$\boldsymbol{(p \wedge q) \rightarrow (q \vee p)}$
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**T**
F	**T**	**F**	**T**	**T**
F	**F**	**F**	**F**	**T**

Prove this is a Tautology: Option 2

$$(p \wedge q) \rightarrow (q \vee p)$$

Use a series of equivalences like so:

$$(p \wedge q) \rightarrow (q \vee p) \equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv \quad \mathbf{T}$$

Associative
- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Distributive
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Absorption
- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

Negation
- $p \vee \neg p \equiv \mathrm{T}$
- $p \wedge \neg p \equiv \mathrm{F}$

Identity
- $p \wedge \mathrm{T} \equiv p$
- $p \vee \mathrm{F} \equiv p$

Domination
- $p \vee \mathrm{T} \equiv \mathrm{T}$
- $p \wedge \mathrm{F} \equiv \mathrm{F}$

Idempotent
- $p \vee p \equiv p$
- $p \wedge p \equiv p$

Commutative
- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

Prove this is a Tautology: Option 2

$$(p \wedge q) \rightarrow (q \vee p)$$

Associative
- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Distributive
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Absorption
- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

Negation
- $p \vee \neg p \equiv \mathrm{T}$
- $p \wedge \neg p \equiv \mathrm{F}$

Use a series of equivalences like so:

$(p \wedge q) \rightarrow (q \vee p) \equiv \neg(p \wedge q) \vee (q \vee p)$	Law of Implication
$\equiv (\neg p \vee \neg q) \vee (q \vee p)$	DeMorgan
$\equiv \neg p \vee (\neg q \vee (q \vee p))$	Associative
$\equiv \neg p \vee ((\neg q \vee q) \vee p)$	Associative
$\equiv \neg p \vee (p \vee (\neg q \vee q))$	Commutative
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	Associative
$\equiv (p \vee \neg p) \vee (q \vee \neg q)$	Commutative (twice)
$\equiv \mathbf{T} \vee \mathbf{T}$	Negation (twice)
$\equiv \mathbf{T}$	Domination/Identity

Identity
- $p \wedge \mathrm{T} \equiv p$
- $p \vee \mathrm{F} \equiv p$

Domination
- $p \vee \mathrm{T} \equiv \mathrm{T}$
- $p \wedge \mathrm{F} \equiv \mathrm{F}$

Idempotent
- $p \vee p \equiv p$
- $p \wedge p \equiv p$

Commutative
- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

Logical Proofs of Equivalence/Tautology

- **Not smaller than truth tables when there are only a few propositional variables…**

- **…but usually *much shorter* than truth table proofs when there are many propositional variables**

- **A big advantage will be that we can extend them to a more in-depth understanding of logic for which truth tables don't apply.**