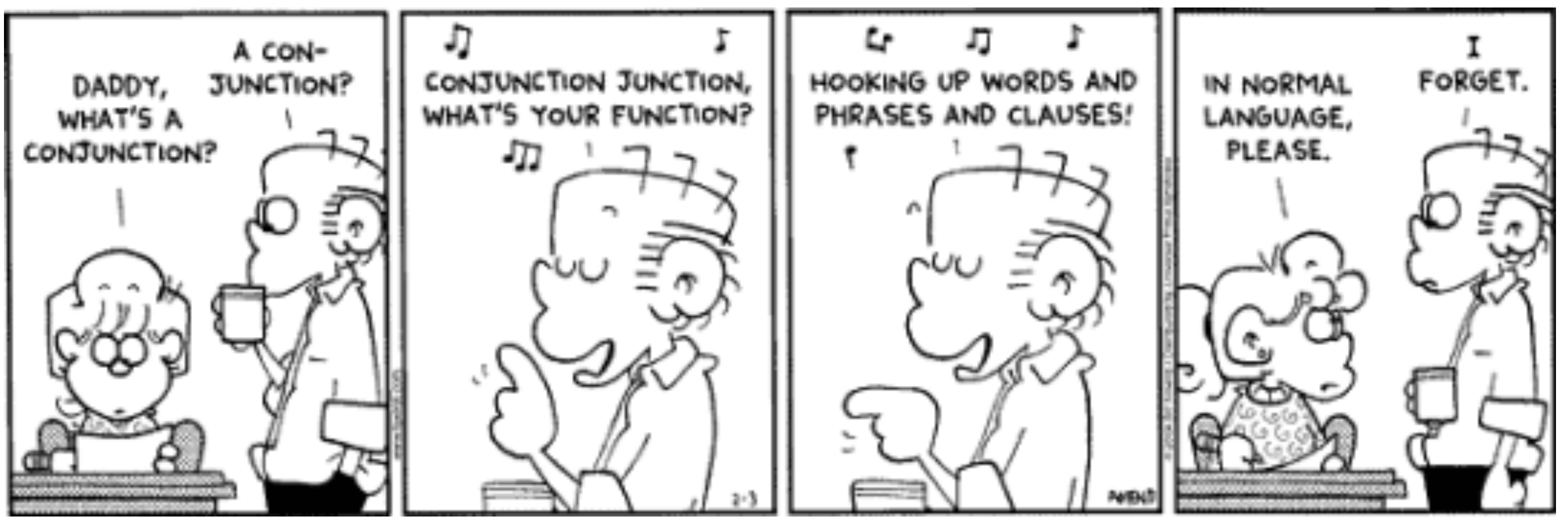
CSE 311: Foundations of Computing

Lecture 2: More Logic, Equivalence & Digital Circuits



Last class: Some Connectives & Truth Tables

Negation (not)

p	$\neg p$
T	F
F	T

Conjunction (and)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (or)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Last class: Implication

"If it's raining, then I have my umbrella"

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \rightarrow q$

(1) "I have collected all 151 Pokémon if I am a Pokémon master"

(2) "I have collected all 151 Pokémon only if I am a Pokémon master"

These sentences are implications in opposite directions:

$p \rightarrow q$

(1) "I have collected all 151 Pokémon if I am a Pokémon master"

(2) "I have collected all 151 Pokémon only if I am a Pokémon master"

These sentences are implications in opposite directions:

(1) "Pokémon masters have all 151 Pokémon"

(2) "People who have 151 Pokémon are Pokémon masters"

So, the implications are:

(1) If I am a Pokémon master, *then* I have collected all 151 Pokémon.

(2) If I have collected all 151 Pokémon, *then* I am a Pokémon master.

$p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication:

- *p* implies *q*
- whenever *p* is true *q* must be true
- if *p* then *q*
- *q* if *p*
- *p* is sufficient for *q*
- *p* only if *q*
- *q is necessary for p*

Biconditional: $p \leftrightarrow q$

- *p* iff *q*
- *p* is equivalent to *q*
- *p* implies *q* and *q* implies *p*
- *p* is necessary and sufficient for *q*

p	q	$p \leftrightarrow q$

Biconditional: $p \leftrightarrow q$

- *p* iff *q*
- *p* is equivalent to *q*
- *p* implies *q* and *q* implies *p*
- *p* is necessary and sufficient for *q*

p	q	$p \leftrightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Back to Garfield...

p "Garfield has black stripes"
q "Garfield is an orange cat"
r "Garfield likes lasagna"

"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"

↓

(p if (q and r)) and (q or (not r))

↓

(p "if" $(q \wedge r)) \wedge$ **(q** $\vee \neg$**r)**

Back to Garfield…

p	“Garfield has black stripes”
q	“Garfield is an orange cat”
r	“Garfield likes lasagna”

“Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna”

↓

(p if (q and r)) and (q or (not r))

↓

(p “if” (q ∧ r)) ∧ **(q** ∨ ¬**r)**

↓

((q ∧ r) ⟶ **p)** ∧ **(q** ∨ ¬**r)**

Analyzing the Garfield Sentence with a Truth Table

p	q	r	$\neg r$	$q \vee \neg r$	$q \wedge r$	$(q \wedge r) \rightarrow p$	$((q \wedge r) \rightarrow p) \wedge (q \vee \neg r)$
F	F	F					
F	F	T					
F	T	F					
F	T	T					
T	F	F					
T	F	T					
T	T	F					
T	T	T					

Analyzing the Garfield Sentence with a Truth Table

p	q	r	$\neg r$	$q \vee \neg r$	$q \wedge r$	$(q \wedge r) \rightarrow p$	$((q \wedge r) \rightarrow p) \wedge (q \vee \neg r)$
F	F	F	T	T	F	T	T
F	F	T	F	F	F	T	F
F	T	F	T	T	F	T	T
F	T	T	F	T	T	F	F
T	F	F	T	T	F	T	T
T	F	T	F	F	F	T	F
T	T	F	T	T	F	T	T
T	T	T	F	T	T	T	T

Converse, Contrapositive

Implication:

$p \rightarrow q$

Converse:

$q \rightarrow p$

Contrapositive:

$\neg q \rightarrow \neg p$

Inverse:

$\neg p \rightarrow \neg q$

Consider

p: x is divisible by 2

q: x is divisible by 4

$p \rightarrow q$	
$q \rightarrow p$	
$\neg q \rightarrow \neg p$	
$\neg p \rightarrow \neg q$	

Converse, Contrapositive

Implication:

$p \rightarrow q$

Contrapositive:

$\neg q \rightarrow \neg p$

Converse:

$q \rightarrow p$

Inverse:

$\neg p \rightarrow \neg q$

<u>Consider</u>

p: x is divisible by 2

q: *x* is divisible by 4

$p \rightarrow q$	
$q \rightarrow p$	
$\neg q \rightarrow \neg p$	
$\neg p \rightarrow \neg q$	

	Divisible By 2	Not Divisible By 2
Divisible By 4		
Not Divisible By 4		

Converse, Contrapositive

Implication:

$p \rightarrow q$

Converse:

$q \rightarrow p$

Contrapositive:

$\neg q \rightarrow \neg p$

Inverse:

$\neg p \rightarrow \neg q$

Consider

p: x is divisible by 2

q: x is divisible by 4

$p \rightarrow q$	
$q \rightarrow p$	
$\neg q \rightarrow \neg p$	
$\neg p \rightarrow \neg q$	

	Divisible By 2	Not Divisible By 2
Divisible By 4	4,8,12,...	Impossible
Not Divisible By 4	2,6,10,...	1,3,5,...

Converse, Contrapositive

Implication:

$p \to q$

Converse:

$q \to p$

Contrapositive:

$\neg q \to \neg p$

Inverse:

$\neg p \to \neg q$

How do these relate to each other?

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T						
T	F						
F	T						
F	F						

Converse, Contrapositive

Implication:

$p \rightarrow q$

Converse:

$q \rightarrow p$

Contrapositive:

$\neg q \rightarrow \neg p$

Inverse:

$\neg p \rightarrow \neg q$

An implication and it's contrapositive have the same truth value!

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \vee \neg p$

$p \oplus p$

$(p \to q) \wedge p$

Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \lor \neg p$

This is a tautology. It's called the "law of the excluded middle".
If p is true, then $p \lor \neg p$ is true. If p is false, then $p \lor \neg p$ is true.

$p \oplus p$

This is a contradiction. It's always false no matter what truth value p takes on.

$(p \rightarrow q) \land p$

This is a contingency. When p=T, q=T, $(T \rightarrow T) \land T$ is true.
When p=T, q=F, $(T \rightarrow F) \land T$ is false.

Logical Equivalence

A = B means **A** and **B** are identical "strings":

- $p \wedge q = p \wedge q$

- $p \wedge q \neq q \wedge p$

Logical Equivalence

A = B means **A** and **B** are identical "strings":

- $p \wedge q = p \wedge q$

  These are equal, because they are character-for-character identical.

- $p \wedge q \neq q \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

A ≡ B means **A** and **B** have identical truth values:

- $p \wedge q \equiv p \wedge q$

- $p \wedge q \equiv q \wedge p$

- $p \wedge q \not\equiv q \vee p$

Logical Equivalence

A = B means A and B are identical "strings":

- $p \wedge q = p \wedge q$

  These are equal, because they are character-for-character identical.

- $p \wedge q \neq q \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

A ≡ B means A and B have identical truth values:

- $p \wedge q \equiv p \wedge q$

  Two formulas that are **equal** also are equivalent.

- $p \wedge q \equiv q \wedge p$

  These two formulas have the same truth table!

- $p \wedge q \not\equiv q \vee p$

  When p=T and q=F, $p \wedge q$ is false, but $p \vee q$ is true!

$A \leftrightarrow B$ vs. $A \equiv B$

$A \equiv B$ is an **assertion over all possible truth values** that A and B always have the same truth values.

$A \leftrightarrow B$ is a **proposition** that may be true or false depending on the truth values of the variables in A and B.

$A \equiv B$ and $(A \leftrightarrow B) \equiv \mathbf{T}$ have the same meaning.

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Negate the statement:

"My code compiles or there is a bug."

To negate the statement,

ask "when is the original statement false".

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Negate the statement:

"My code compiles or there is a bug."

To negate the statement,

ask "when is the original statement false".

It's false when not(my code compiles) AND not(there is a bug).

Translating back into English, we get:

My code doesn't compile and there is not a bug.

De Morgan's Laws

Example: $\neg(p \land q) \equiv (\neg p \lor \neg q)$

p	q	$\neg p$	$\neg q$	$\neg p \lor \neg q$	$p \land q$	$\neg(p \land q)$	$\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)$
T	T						
T	F						
F	T						
F	F						

De Morgan's Laws

Example: $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
T	T	F	F	F	T	F	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

```
if (!(front != null && value > front.data))
    front = new ListNode(value, front);
else {
    ListNode current = front;
    while (current.next != null && current.next.data < value))
        current = current.next;
    current.next = new ListNode(value, current.next);
}
```

De Morgan's Laws

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

```
!(front != null && value > front.data)
```

$$\equiv$$

```
front == null || value <= front.data
```

You've been using these for a while!

Law of Implication

$$p \to q \equiv \neg p \lor q$$

p	q	$p \to q$	$\neg p$	$\neg p \lor q$	$p \to q \leftrightarrow \neg p \lor q$
T	T				
T	F				
F	T				
F	F				

Law of Implication

$$p \to q \equiv \neg p \lor q$$

p	q	$p \to q$	$\neg p$	$\neg p \lor q$	$p \to q \leftrightarrow \neg p \lor q$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Some Equivalences Related to Implication

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

Properties of Logical Connectives

We will always give you this list!

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$