CSE 311: Foundations of Computing I

Homework 4 (due Wednesday, May 1 at 11:00 PM)

Directions: *Write up carefully argued solutions to the following problems. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. However, you may use results from lecture, the theorems handout, and previous homeworks without proof.*

Note: Problems asking you to "prove" a claim require an English (not formal) proof. Within an English proof, however, it is okay to write out a chain of equivalences in table form, with explanations for each substitution along the side, as in previous assignments, rather than writing those steps with sentences.

1. Starting Line (16 points)

Let the domain of discourse be the real numbers ($\mathbb{R}$). We define the predicate $\mathrm{OnLine}(a, b, x, y)$ to be true iff (x, y) lies on the line with slope a and intercept b (i.e. iff $ax+b = y$) and the predicate $\mathrm{InRectangle}(u, v, r, s, x, y)$ to be true iff (x, y) lies inside an $r \times s$ rectangle, which has its bottom left corner at (u, v).

Prove the following claim:

$$\begin{aligned}\forall u.\forall v.\forall a.\forall r.(a > 0 \land r > 0) \quad \rightarrow \quad & \exists s.\exists b_1.\exists b_2.\exists x\ \exists y.\ s > 0 \land x \neq u \land y \neq v \land \\ & \mathrm{InRectangle}(u, v, r, s, x, y) \land \\ & \mathrm{OnLine}(a, b_1, u, v) \land \mathrm{OnLine}(a, b_1, u + r, v + s) \land \mathrm{OnLine}(a, b_1, x, y) \land \\ & \mathrm{OnLine}(-a, b_2, u, v + s) \land \mathrm{OnLine}(-a, b_2, u + r, v) \land \mathrm{OnLine}(-a, b_2, x, y)\end{aligned}$$

You should not need to cite any additional facts about geometry, only the definitions given above and possibly some simple algebra. In addition, here are some hints:

- To prove that something exists, you can simply demonstrate an object with that property. For example, if we are asked to prove that $\exists x\,(x > 0)$, we can simply note that $1 > 0$. (Formally, the first claim follows from the second by "Intro $\exists$", but that step does not need to be explained in an English proof.)
- You should be able to write down the first two sentences of the proof *right now*...

2. All Sets Are Off (18 points)

Let R, S, and T be sets. Prove or disprove the following claims.

(a) [8 Points] $(R \setminus S) \setminus T = R \setminus (S \setminus T)$

(b) [10 Points] $(R \setminus S) \setminus T = R \setminus (S \cup T)$

3. Power Nap (16 points)

Let S and T be sets. Prove or disprove the following claims.

(a) [8 Points] $\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T)$

(b) [8 Points] $\mathcal{P}(S \cap T) = \overline{(\overline{\mathcal{P}(S)} \cap \overline{\mathcal{P}(T)})}$, where complements are taken in a universe of $\mathcal{P}(S \cup T)$.

4. Keeping Up With the Cartesians (18 points)

Let A, B, and C be sets. Suppose that $A \times B \subseteq B \times C$ and $A, B \neq \emptyset$.

(a) [8 Points] Using the assumptions above, prove that $A \subseteq B$.

(b) [8 Points] Using the assumptions above, prove that $B \subseteq C$.

(c) [2 Points] Use parts (a) and (b) to prove that $A \subseteq C$.

5. Weekend At Cape Mod (12 points)

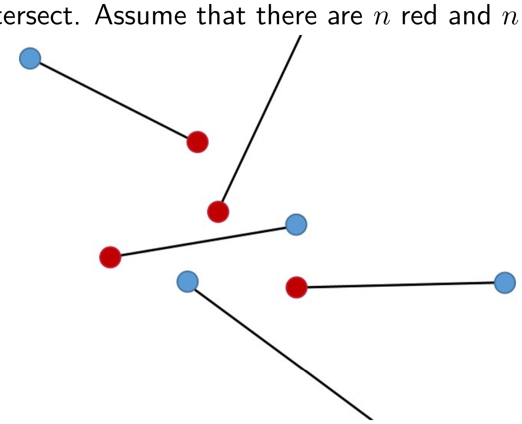
Let a, b, c, and m be integers with c and m both positive. Prove that $a \equiv b \pmod{m}$ holds if and only if $ca \equiv cb \pmod{cm}$ does.

6. A Good Prime Was Had By All (20 points)

Let $p > 3$. Prove that, if p is prime, then $p \equiv 1 \pmod 6$ or $p \equiv 5 \pmod 6$.
(**Hint**: consider the proof strategies discussed in lecture.)

7. Extra Credit: Matchmaker, Matchmaker, Make Me a Match (0 points)

In this problem, you will show that given n red points and n blue points in the plane such that no three points lie on a common line, it is possible to draw line segments between red-blue pairs so that all the pairs are matched and none of the line segments intersect. Assume that there are n red and n blue points fixed in the plane.



A *matching* M is a collection of n line segments connecting distinct red-blue pairs. The *total length* of a matching M is the sum of the lengths of the line segments in M. Say that a matching M is *minimal* if there is no matching with a smaller total length.

Let IsMinimal(M) be the predicate that is true precisely when M is a minimal matching. Let HasCrossing(M) be the predicate that is true precisely when there are two line segments in M that cross each other.

Give an argument in English explaining why there must be at least one matching M so that IsMinimal(M) is true, i.e.

$$\exists M \text{IsMinimal}(M))$$

Give an argument in English explaining why

$$\forall M (\text{HasCrossing}(M) \rightarrow \neg \text{IsMinimal}(M))$$

Now use the two results above to give a proof of the statement:

$$\exists M \neg \text{HasCrossing}(M).$$