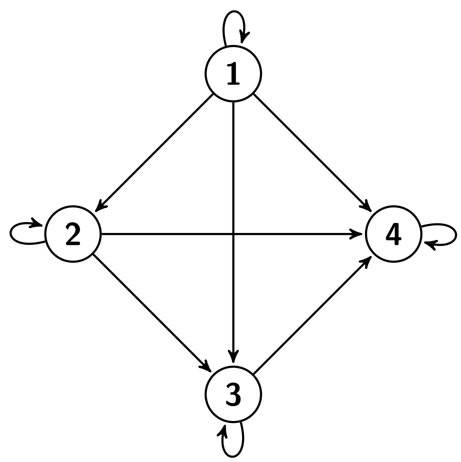
CSE 311: Foundations of Computing I

Section 9: Relations and DFAs Solutions

1. Relations

(a) Draw the transitive-reflexive closure of $\{(1,2),(2,3),(3,4)\}$.

Solution:



(b) Suppose that R is reflexive. Prove that $R \subseteq R^2$.

Solution:

Let $(a,b) \in R$ be arbitrary. Since R is reflexive, we know $(b,b) \in R$ as well. Since there is a b such that $(a,b) \in R$ and $(b,b) \in R$, it follows that $(a,b) \in R^2$. Since a and b were arbitrary, we've shown $R \subseteq R^2$.

(c) Consider the relation $R = \{(x,y) \ : \ x = y + 1\}$ on $\mathbb{N}$.
Is R reflexive? Transitive? Symmetric? Anti-symmetric?

Solution:

It isn't reflexive, because $1 \neq 1+1$; so, $(1,1) \notin R$. It isn't symmetric, because $(2,1) \in R$ (because $2 = 1+1$), but $(1,2) \notin R$, because $1 \neq 2+1$. It isn't transitive, because note that $(3,2) \in R$ and $(2,1) \in R$, but $(3,1) \notin R$. It is anti-symmetric, because consider $(x,y) \in R$ such that $x \neq y$. Then, $x = y+1$ by definition of R. However, $(y,x) \notin R$, because $y = x - 1 \neq x + 1$.

(d) Consider the relation $S = \{(x,y) \ : \ x^2 = y^2\}$ on $\mathbb{R}$. Prove that S is reflexive, transitive, and symmetric.

Solution:

Let $x \in \mathbb{R}$ be arbitrary. We know that $x^2 = x^2$, so $(x,x) \in R$. Since x was arbitrary, we have shown that R is reflexive.

Let $(x,y) \in R$ be arbitrary. From the definition of R, we know that $x^2 = y^2$. Since we then also have $y^2 = x^2$, we can see that $(y,x) \in R$. Since x and y were arbitrary, we have shown R is symmetric.

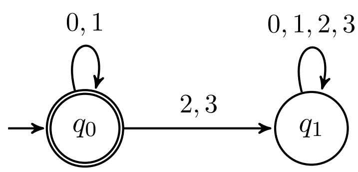
Let $x,y,z \in \mathbb{R}$ be arbitrary. Suppose $(x,y) \in R$ and $(y,z) \in R$. From the definition of R, we have $x^2 = y^2$ and $y^2 = z^2$. Putting these together, we have $x^2 = z^2$, so $(x,z) \in R$. Since x, y, and z were arbitrary, we have shown that R is transitive.

2. DFAs

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1, 2, 3\}$.

(a) All binary strings.

Solution:

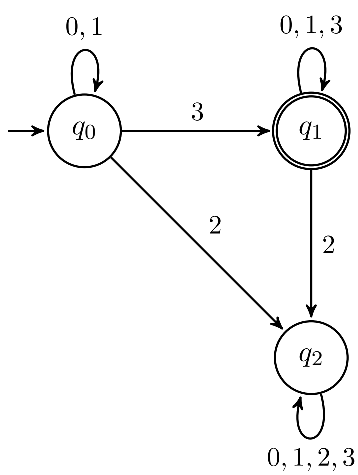


q_0**:** binary strings

q_1**:** strings that contain a character which is not 0 or 1.

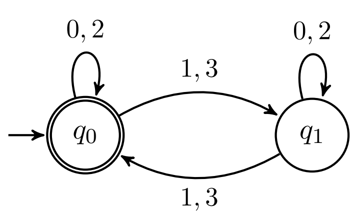
(b) All strings that contain at least one 3 but no 2.

Solution:



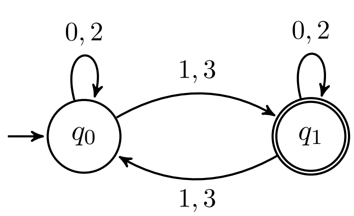
(c) All strings whose digits sum to an even number.

Solution:



(d) All strings whose digits sum to an odd number.

Solution:

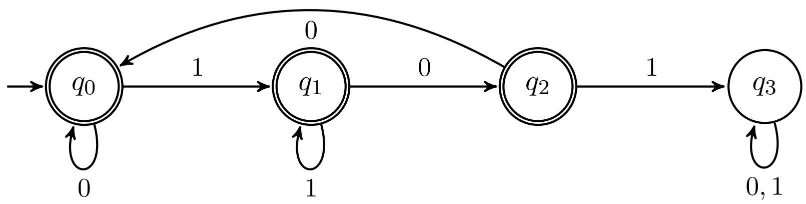


3. DFAs II

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1\}$.

(a) Strings that do not contain the substring 101.

Solution:



q_3**:** string that contain 101.

q_2**:** strings that don't contain 101 and end in 10.

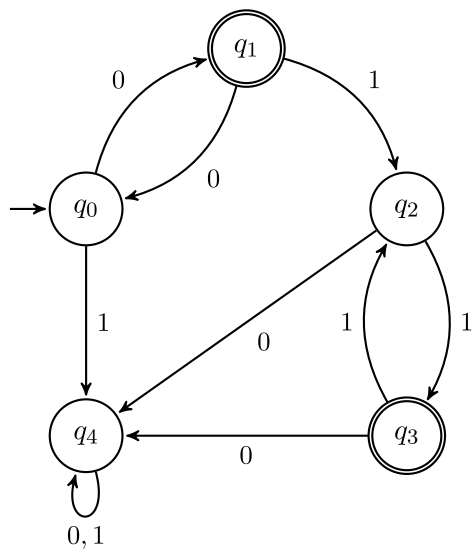
q_1**:** strings that don't contain 101 and end in 1.

q_0**:** ε, 0, strings that don't contain 101 and end in 00.

(b) Strings that contain an even number of 1s and odd number of 0's and do not contain the substring 10.

Solution:

Note that a binary string not containing 10 as a substring must be of the form 0^*1^*. The additional constraints mean we are looking for numbers of the form 0^x1^y with x odd and y even.



q_4**:** strings that are not of the form 0^*1^* or are of the form 0^x11^* with x even

q_3**:** strings of the form 0^x1^y with x odd and y even

q_2**:** strings of the form 0^x1^y with x and y odd

q_1**:** strings of the form 0^x with x odd

q_0**:** strings of the form 0^x with x even

4. Powers of Relations

Let A be a set and R a relation on A. Use induction to prove that R^n is exactly the pairs of elements from A that are connected by a path of length n in the graph $G = (A, R)$.

Solution:

Let $P(n)$ be "for all $a, b \in A$, we have $(a, b) \in R^n$ iff there is a path of length n from a to b in the graph G". We will prove $P(n)$ for all integers $n \in \mathbb{N}$ by induction.

Base Case $(n = 0)$**:** $R^0 = \{(a, a) \mid a \in A\}$ and every element is and can only be connected to itself by a path of length 0, so $P(0)$ holds.

Induction Hypothesis: Assume that $P(k)$ holds for an arbitrary $k \in \mathbb{N}$.

Induction Step: Our goal is to show $P(k + 1)$. I.e., that R^{n+1} is exactly the set of pairs connected by a path of length $n + 1$ in G. Note that, by definition, $R^{n+1} = \{(a, c) \mid \exists b \in A\, ((a, b) \in R^n \wedge (b, c) \in R)\}$.

By the induction hypothesis, $(a, b) \in R^n$ holds iff a and b are connected by a path of length n in G. Hence, if $(a, c) \in R^{n+1}$, then there is some $b \in A$ such that there is a path $a = v_0, v_1, \ldots, v_n = b$ in G, which means that $a = v_0, v_1, \ldots, b, c$ is a path of length $n + 1$ connecting a and c in G.

On the other hand, suppose that $a = v_0, v_1, \ldots, v_n, v_{n+1} = c$ is a path of length $n + 1$ connecting a and c in G. By the definition of a path, this means that $(v_n, c) \in R$. Furthermore, since $a = v_0, v_1, \ldots, v_n$ is a path of length n in G, by the inductive hypothesis, we have $(a, v_n) \in R^n$. Thus, by the definition of R^{n+1}, we can see that $(a, c) \in R^{n+1}$.

Conclusion: $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.