CSE 311: Foundations of Computing I

Section 2: Equivalences and Boolean Algebra Solutions

1. Equivalences

Prove that each of the following pairs of propositional formulae are equivalent using propositional equivalences.

(a) $\neg p \to (q \to r)$ $\qquad$ $q \to (p \lor r)$

Solution:

$\neg p \to (q \to r)$	$\equiv$	$\neg\neg p \lor (q \to r)$	[Law of Implication]
	$\equiv$	$p \lor (q \to r)$	[Double Negation]
	$\equiv$	$p \lor (\neg q \lor r)$	[Law of Implication]
	$\equiv$	$(p \lor \neg q) \lor r$	[Associativity]
	$\equiv$	$(\neg q \lor p) \lor r$	[Commutativity]
	$\equiv$	$\neg q \lor (p \lor r)$	[Associativity]
	$\equiv$	$q \to (p \lor r)$	[Law of Implication]

(b) $p \leftrightarrow q$ $\qquad$ $(p \land q) \lor (\neg p \land \neg q)$

Solution:

$p \leftrightarrow q$	$\equiv$	$(p \to q) \land (q \to p)$	[iff is two implications]
	$\equiv$	$(\neg p \lor q) \land (q \to p)$	[Law of Implication]
	$\equiv$	$(\neg p \lor q) \land (\neg q \lor p)$	[Law of Implication]
	$\equiv$	$((\neg p \lor q) \land \neg q) \lor ((\neg p \lor q) \land p)$	[Distributivity]
	$\equiv$	$(\neg q \land (\neg p \lor q)) \lor ((\neg p \lor q) \land p)$	[Commutativity]
	$\equiv$	$((\neg q \land \neg p) \lor (\neg q \land q)) \lor ((\neg p \lor q) \land p)$	[Distributivity]
	$\equiv$	$((\neg p \land \neg q) \lor (\neg q \land q)) \lor ((\neg p \lor q) \land p)$	[Commutativity]
	$\equiv$	$((\neg p \land \neg q) \lor (q \land \neg q)) \lor ((\neg p \lor q) \land p)$	[Commutativity]
	$\equiv$	$((\neg p \land \neg q) \lor (q \land \neg q)) \lor (p \land (\neg p \lor q))$	[Commutativity]
	$\equiv$	$((\neg p \land \neg q) \lor (q \land \neg q)) \lor ((p \land \neg p) \lor (p \land q))$	[Distributivity]
	$\equiv$	$((\neg p \land \neg q) \lor F) \lor ((p \land \neg p) \lor (p \land q))$	[Negation]
	$\equiv$	$((\neg p \land \neg q) \lor F) \lor (F \lor (p \land q))$	[Negation]
	$\equiv$	$(\neg p \land \neg q) \lor (F \lor (p \land q))$	[Identity]
	$\equiv$	$(\neg p \land \neg q) \lor ((p \land q) \lor F)$	[Commutativity]
	$\equiv$	$(\neg p \land \neg q) \lor (p \land q)$	[Identity]
	$\equiv$	$(p \land q) \lor (\neg p \land \neg q)$	[Commutativity]

2. Non-equivalence

Prove that the following pairs of propositional formulae are not equivalent by finding inputs they differ on.

(a) $p \to q$ $\qquad\qquad$ $q \to p$

Solution:

A: They differ when $p = \mathsf{T}$ and $q = \mathsf{F}$.

B: They differ when $p = \mathsf{T}$ and $q = \mathsf{F}$ since $p \to q = \mathsf{T} \to \mathsf{F} \equiv \mathsf{F}$ but $q \to p \equiv \mathsf{F} \to \mathsf{T} \equiv \mathsf{T}$.

(b) $p \to (q \wedge r)$ $\qquad\qquad$ $(p \to q) \wedge r$

Solution:

A: They differ when $p = r = \mathsf{F}$ since $p \to (q \wedge r) = \mathsf{F} \to (q \wedge \mathsf{F}) \equiv \mathsf{T}$ but $(p \to q) \wedge r = (\mathsf{F} \to q) \wedge \mathsf{F} \equiv \mathsf{F}$.

B: They differ when $p = r = \mathsf{F}$ since $p \to (q \wedge r) = \mathsf{F} \to (q \wedge \mathsf{F}) \equiv \mathsf{F} \to \mathsf{F} \equiv \mathsf{T}$ and on the other hand $(p \to q) \wedge r = (\mathsf{F} \to q) \wedge \mathsf{F} \equiv \mathsf{T} \wedge \mathsf{F} \equiv \mathsf{F}$.

3. Boolean Algebra

For each of the following parts, write the logical expression using boolean algebra operators. Then, simplify it using axioms and theorems of boolean algebra.

(a) $\neg p \vee (\neg q \vee (p \wedge q))$

Solution:

A: First, we replace $\neg, \vee$, and $\wedge$. This gives us $p' + q' + pq$. (Note that the parentheses are not necessary in boolean algebra since the operations are associative.) Next, we can use DeMorgan's laws to get the slightly simpler $(pq)' + pq$. Then, we can use commutativity to get $pq + (pq)'$ and complementarity to get 1. (Note that this is another way of saying the formula is a tautology.)

B: Replacing $\neg$, $\vee$, and $\wedge$ gives us $p' + q' + pq$. (Note that parentheses are not necessary in boolean algebra since the operations are associative.) Then, we can simplify as follows:

$$\begin{aligned} p' + q' + pq &= (pq)' + pq && \text{De Morgan} \\ &= pq + (pq)' && \text{Commutativity} \\ &= 1 && \text{Complementarity} \end{aligned}$$

Since 1 in boolean algebra means T, this shows that the original expression is a tautology.

(b) $\neg(p \vee (q \wedge p))$

Solution:

A:

$$\begin{aligned} (p + qp)' &= p'(qp)' && \text{De Morgan} \\ &= p'(q' + p') && \text{De Morgan (on second term)} \\ &= p'(p' + q') && \text{Commutativity} \\ &= p' && \text{Absorption} \end{aligned}$$

B: Translating to Boolean algebra, we get $(p + qp)'$. Then, we can simplify as follows:

$$\begin{aligned}
(p + qp)' &= p'(qp)' && \text{De Morgan} \\
&= p'(q' + p') && \text{De Morgan} \\
&= p'(p' + q') && \text{Commutativity} \\
&= p' && \text{Absorption}
\end{aligned}$$

4. Properties of XOR

Like $\wedge$ and $\vee$, the $\oplus$ operator (exclusive or) has many interesting properties. For example, it is easy to verify with a truth table that $\oplus$ is also associative. In this problem, we will prove some additional properties of $\oplus$.

Use equivalence chains to prove each of the facts stated below. For this problem only, you may also use the equivalence

$$p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$$

which you may cite as "Definition of $\oplus$". This equivalence allows you to translate $\oplus$ into an expression involving only $\wedge$, $\vee$, and $\neg$, so that the standard equivalences can then be applied.

(a) $p \oplus q \equiv q \oplus p$ (Commutativity)

Solution:

$$\begin{aligned}
p \oplus q &\equiv (p \wedge \neg q) \vee (\neg p \wedge q) && \text{Definition of } \oplus \\
&\equiv (\neg p \wedge q) \vee (p \wedge \neg q) && \text{Commutativity} \\
&\equiv (q \wedge \neg p) \vee (\neg q \wedge p) && \text{Commutativity} \\
&\equiv q \oplus p && \text{Definition of } \oplus
\end{aligned}$$

(b) $p \oplus p \equiv \mathsf{F}$ and $p \oplus \neg p \equiv \mathsf{T}$

Solution:

$$\begin{aligned}
p \oplus p &\equiv (p \wedge \neg p) \vee (\neg p \wedge p) && \text{Definition of } \oplus \\
&\equiv (p \wedge \neg p) \vee (p \wedge \neg p) && \text{Commutativity} \\
&\equiv (p \wedge \neg p) && \text{Idempotency} \\
&\equiv \mathsf{F} && \text{Negation}
\end{aligned}$$

$$\begin{aligned}
p \oplus \neg p &\equiv (p \wedge \neg\neg p) \vee (\neg p \wedge \neg p) && \text{Definition of } \oplus \\
&\equiv (p \wedge p) \vee (\neg p \wedge \neg p) && \text{Double Negation} \\
&\equiv p \vee \neg p && \text{Idempotency} \\
&\equiv \mathsf{T} && \text{Negation}
\end{aligned}$$

(c) $p \oplus \mathsf{F} \equiv p$ and $p \oplus \mathsf{T} \equiv \neg p$

Solution:

$p \oplus \mathsf{F} \equiv (p \wedge \neg\mathsf{F}) \vee (\neg p \wedge \mathsf{F})$	Definition of $\oplus$
$\equiv (p \wedge (\neg\mathsf{F} \vee \mathsf{F})) \vee (\neg p \wedge \mathsf{F})$	Identity
$\equiv (p \wedge (\mathsf{F} \vee \neg\mathsf{F})) \vee (\neg p \wedge \mathsf{F})$	Commutativity
$\equiv (p \wedge \mathsf{T}) \vee (\neg p \wedge \mathsf{F})$	Negation
$\equiv p \vee (\neg p \wedge \mathsf{F})$	Identity
$\equiv p \vee \mathsf{F}$	Domination
$\equiv p$	Identity

$p \oplus \mathsf{T} \equiv (p \wedge \neg\mathsf{T}) \vee (\neg p \wedge \mathsf{T})$	Definition of $\oplus$
$\equiv (p \wedge \neg\mathsf{T}) \vee \neg p$	Identity
$\equiv (\neg\neg p \wedge \neg\mathsf{T}) \vee \neg p$	Double Negation
$\equiv \neg(\neg p \vee \mathsf{T}) \vee \neg p$	De Morgan
$\equiv \neg\mathsf{T} \vee \neg p$	Domination
$\equiv \neg(\mathsf{T} \wedge p)$	De Morgan
$\equiv \neg(p \wedge \mathsf{T})$	Commutativity
$\equiv \neg p$	Identity

(d) $(\neg p) \oplus q \equiv \neg(p \oplus q) \equiv p \oplus (\neg q)$. I.e., negating one of the inputs negates the overall expression.

Solution:

$\neg(p \oplus q) \equiv \neg((p \wedge \neg q) \vee (\neg p \wedge q))$	Definition of $\oplus$
$\equiv \neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q)$	De Morgan
$\equiv (\neg p \vee \neg\neg q) \wedge (\neg\neg p \vee \neg q)$	De Morgan
$\equiv (\neg p \vee q) \wedge (p \vee \neg q)$	Double Negation
$\equiv ((\neg p \vee q) \wedge p) \vee ((\neg p \vee q) \wedge \neg q)$	Distributivity
$\equiv (p \wedge (\neg p \vee q)) \vee (\neg q \wedge (\neg p \vee q))$	Commutativity
$\equiv ((p \wedge \neg p) \vee (p \wedge q)) \vee ((\neg q \wedge \neg p) \vee (\neg q \wedge q))$	Distributivity
$\equiv ((p \wedge \neg p) \vee (p \wedge q)) \vee ((\neg q \wedge \neg p) \vee (q \wedge \neg q))$	Commutativity
$\equiv (\mathsf{F} \vee (p \wedge q)) \vee ((\neg q \wedge \neg p) \vee \mathsf{F})$	Negation
$\equiv ((p \wedge q) \vee \mathsf{F}) \vee ((\neg q \wedge \neg p) \vee \mathsf{F})$	Commutativity
$\equiv (p \wedge q) \vee (\neg q \wedge \neg p)$	Negation
$\equiv (\neg\neg p \wedge q) \vee (\neg q \wedge \neg p)$	Double Negation
$\equiv (\neg\neg p \wedge q) \vee (\neg p \wedge \neg q)$	Commutativity
$\equiv (\neg p \wedge \neg q) \vee (\neg\neg p \wedge q)$	Commutativity
$\equiv \neg p \oplus q$	Definition of $\oplus$

The second equivalence $\neg(p \oplus q) \equiv p \oplus (\neg q)$ follows from the first and Commutativity (part a).