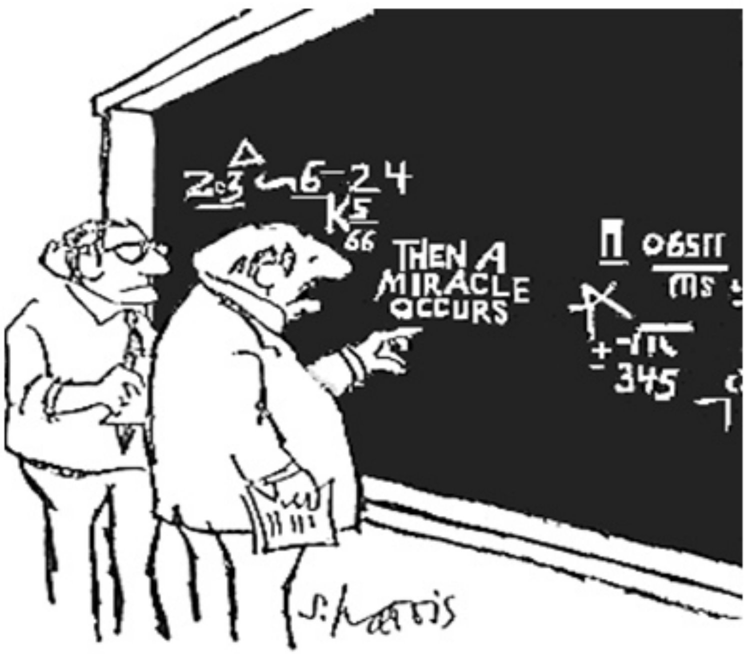
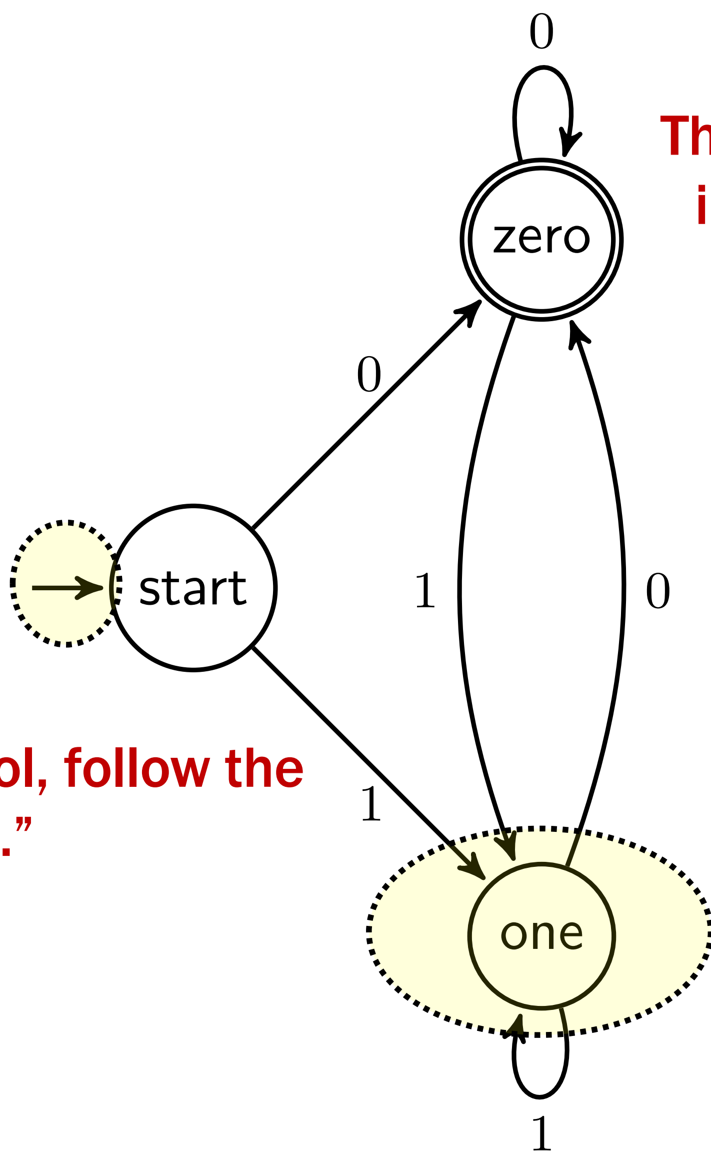
CSE 311: Foundations of Computing

Lecture 23: Finite State Machine Minimization & NFAs



" I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Finite State Machines



The "double circle" means "the input is good if it ends here"

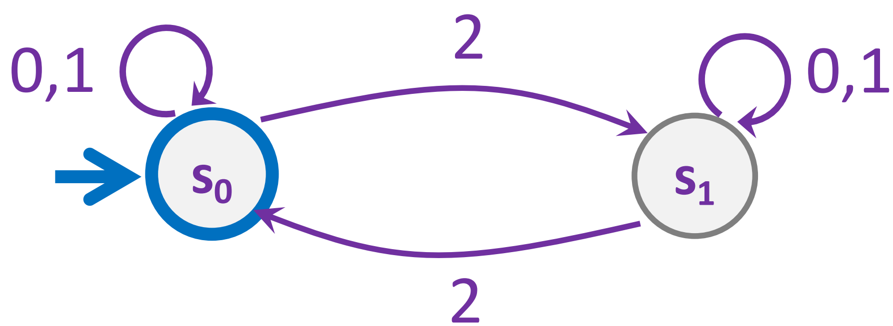
"Start here"

"If I get this symbol, follow the arrow…"

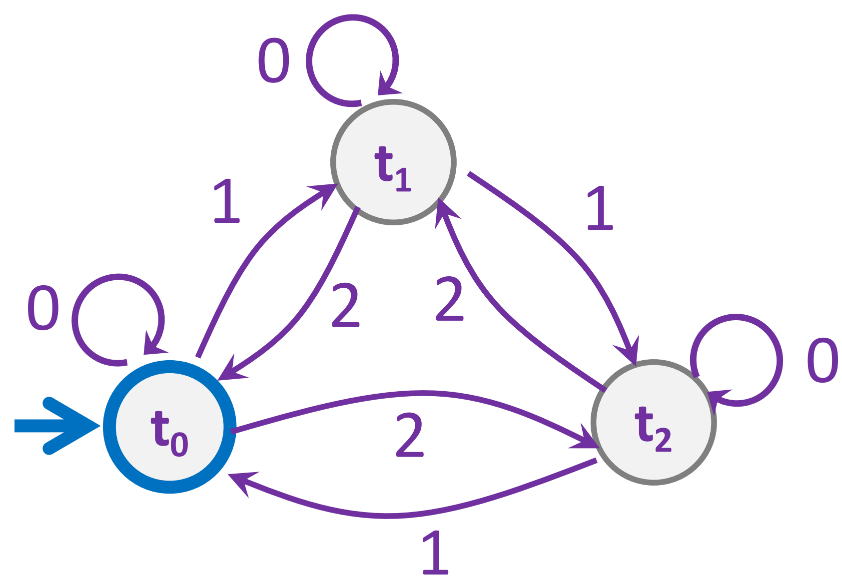
The circles are called "states"
We're only in a single state at any point in time…

Strings over {0, 1, 2}

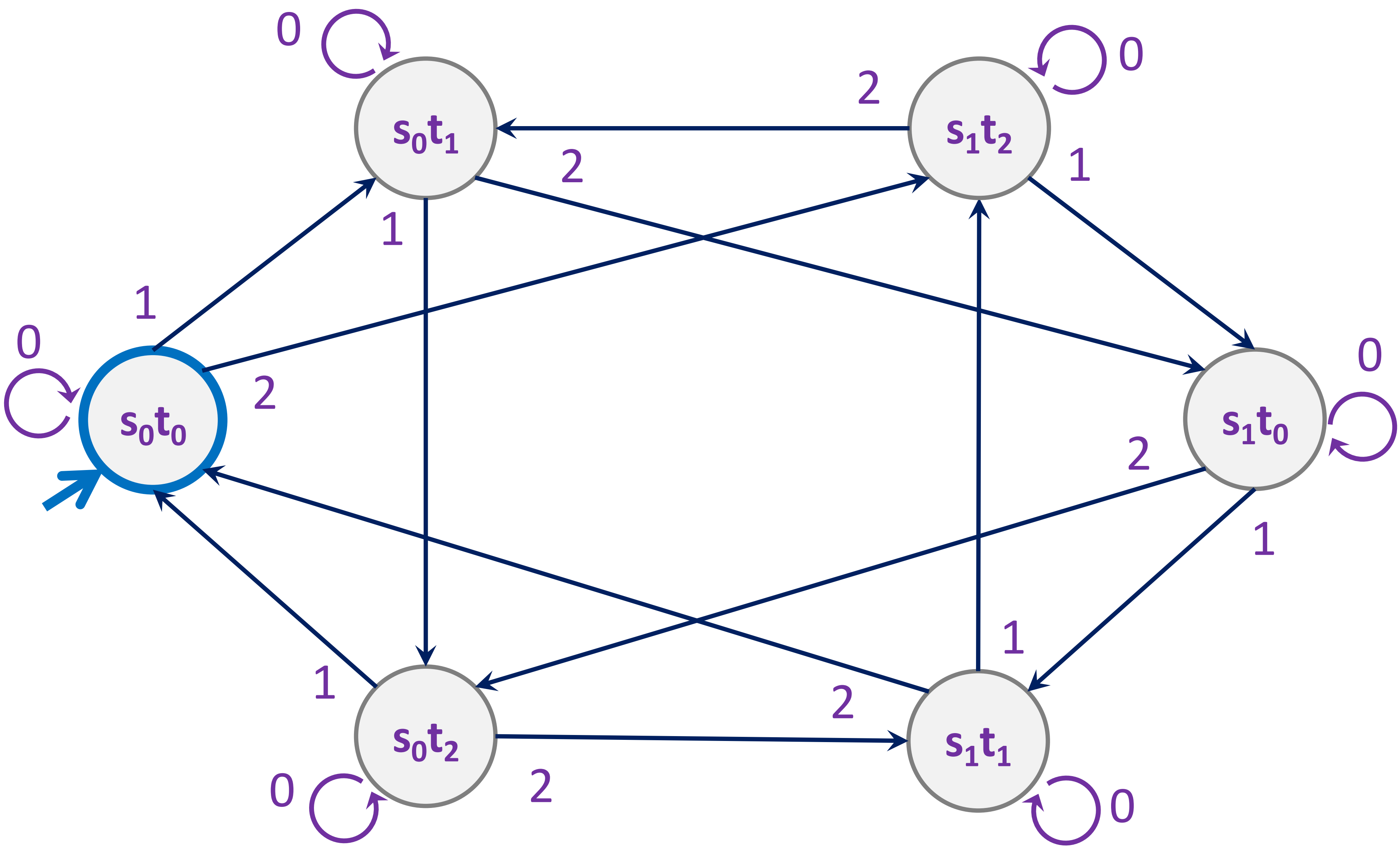
M_1: Strings with an even number of 2's



M_2: Strings where the sum of digits mod 3 is 0

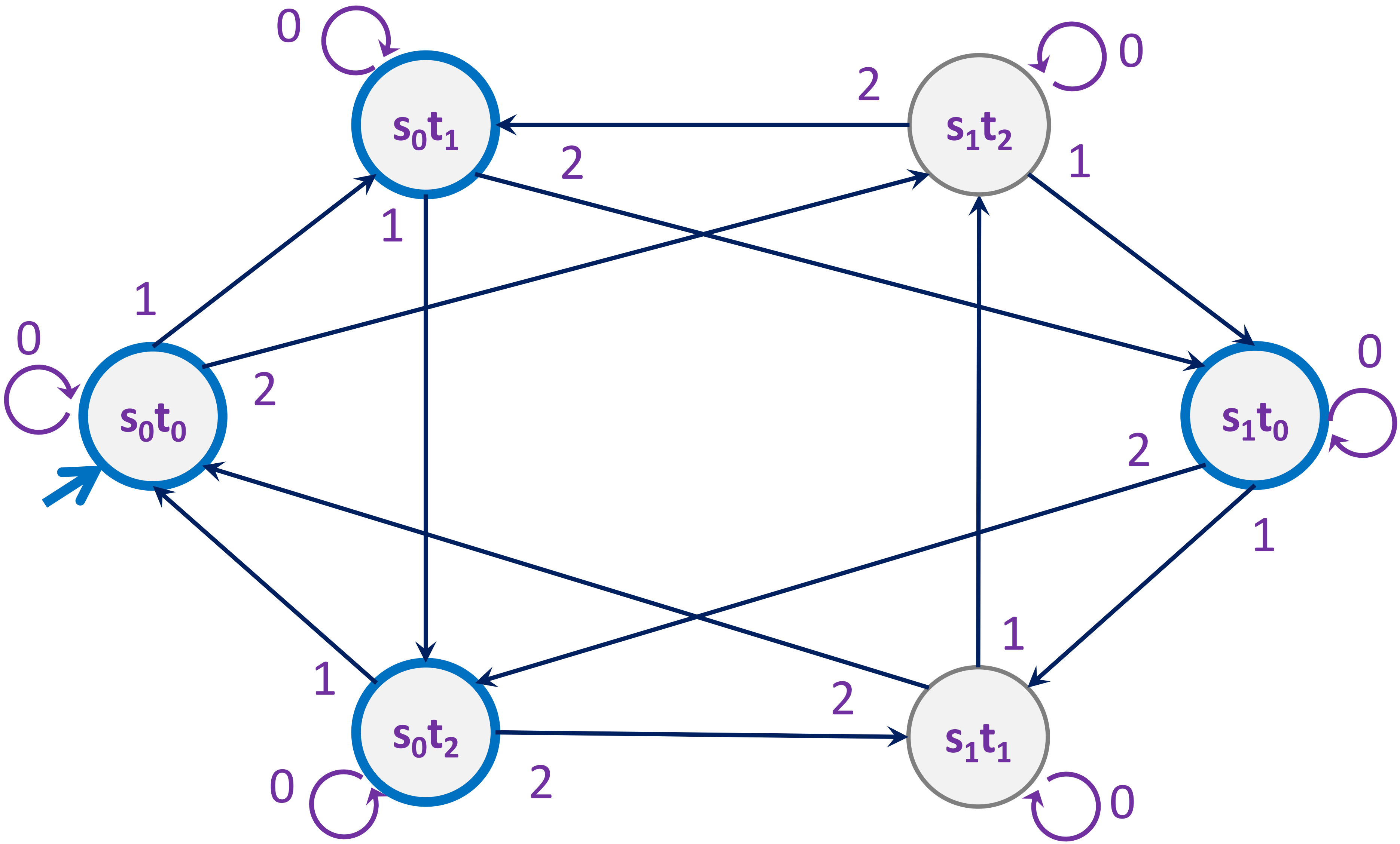


Strings over {0,1,2} w/ even number of 2's and mod 3 sum 0



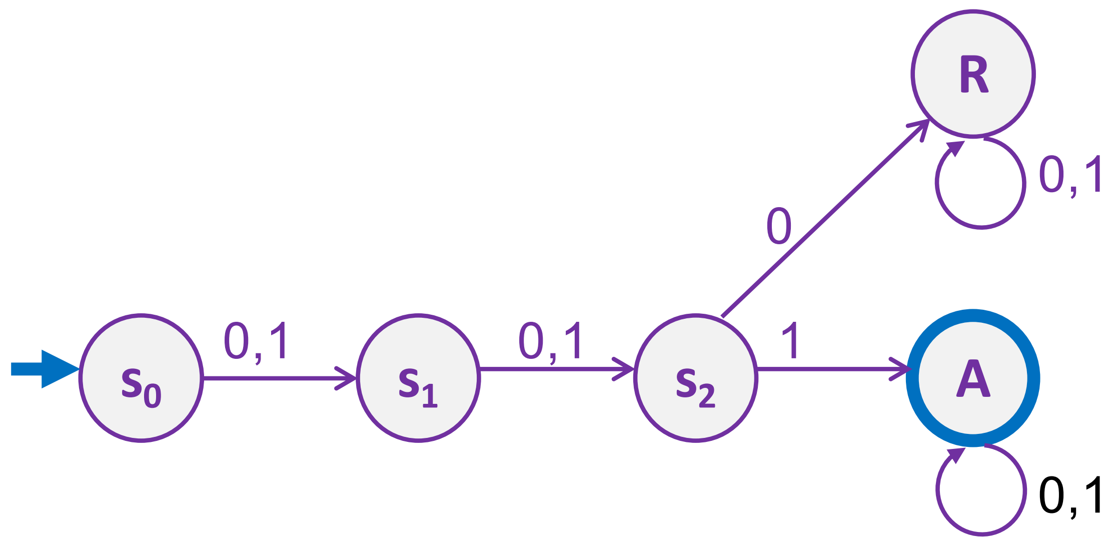
Strings over {0,1,2} **w/ even number of** 2’**s or mod** 3 **sum** 0

Strings over {0,1,2} w/ even number of 2's or mod 3 sum 0



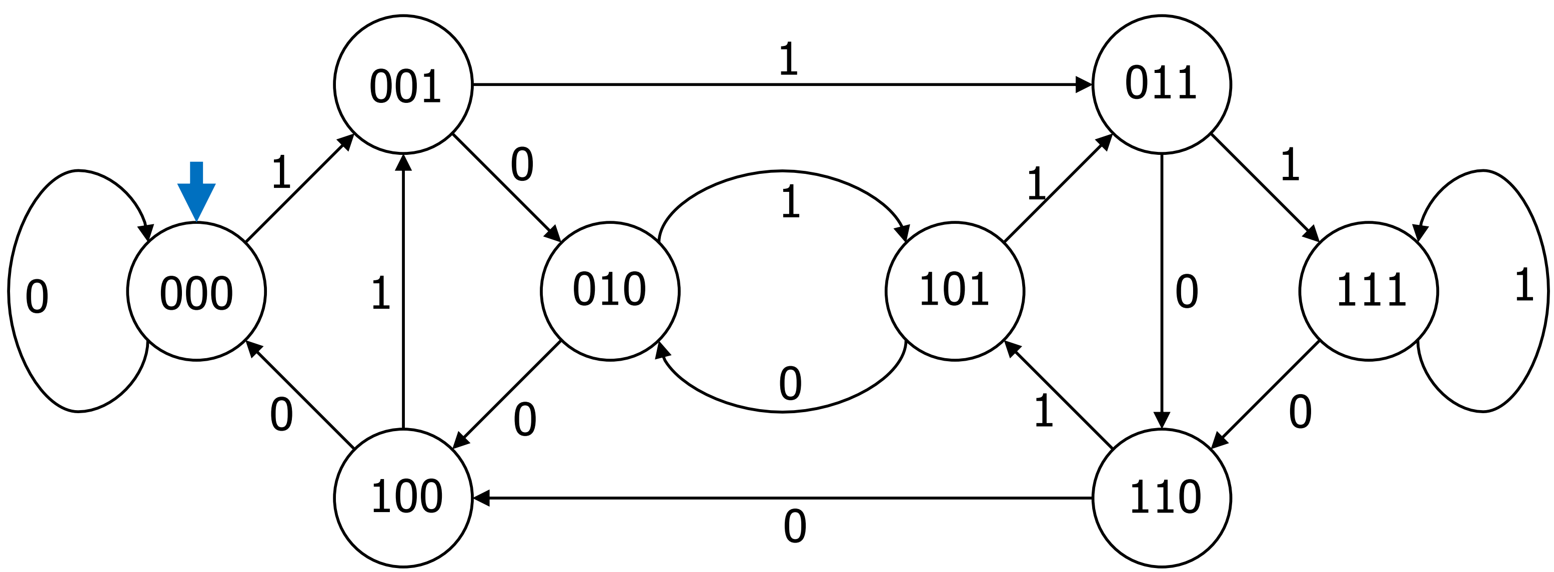
The set of binary strings with a 1 in the 3rd position from the start

The set of binary strings with a 1 in the 3rd position from the start

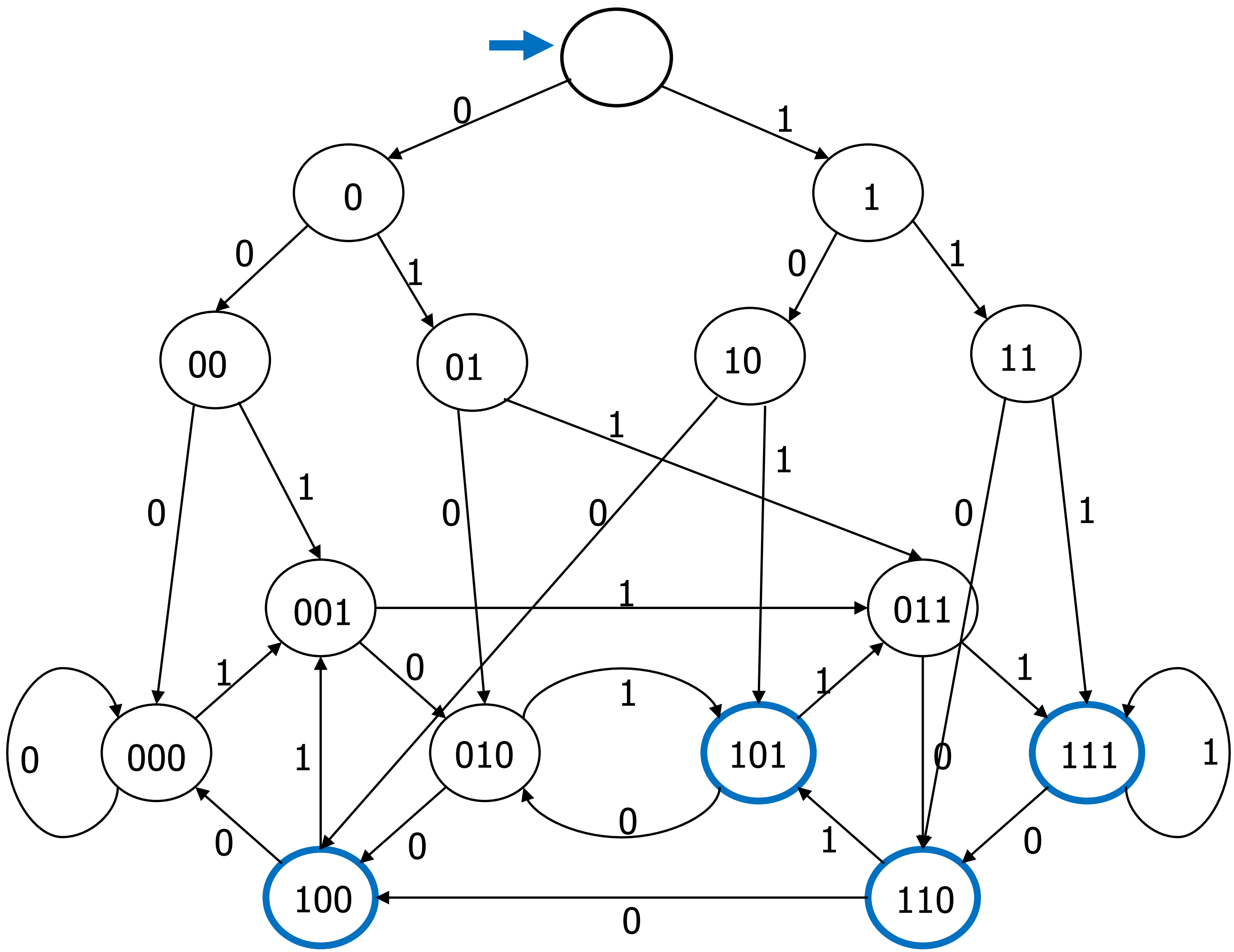


The set of binary strings with a 1 in the 3rd position from the end

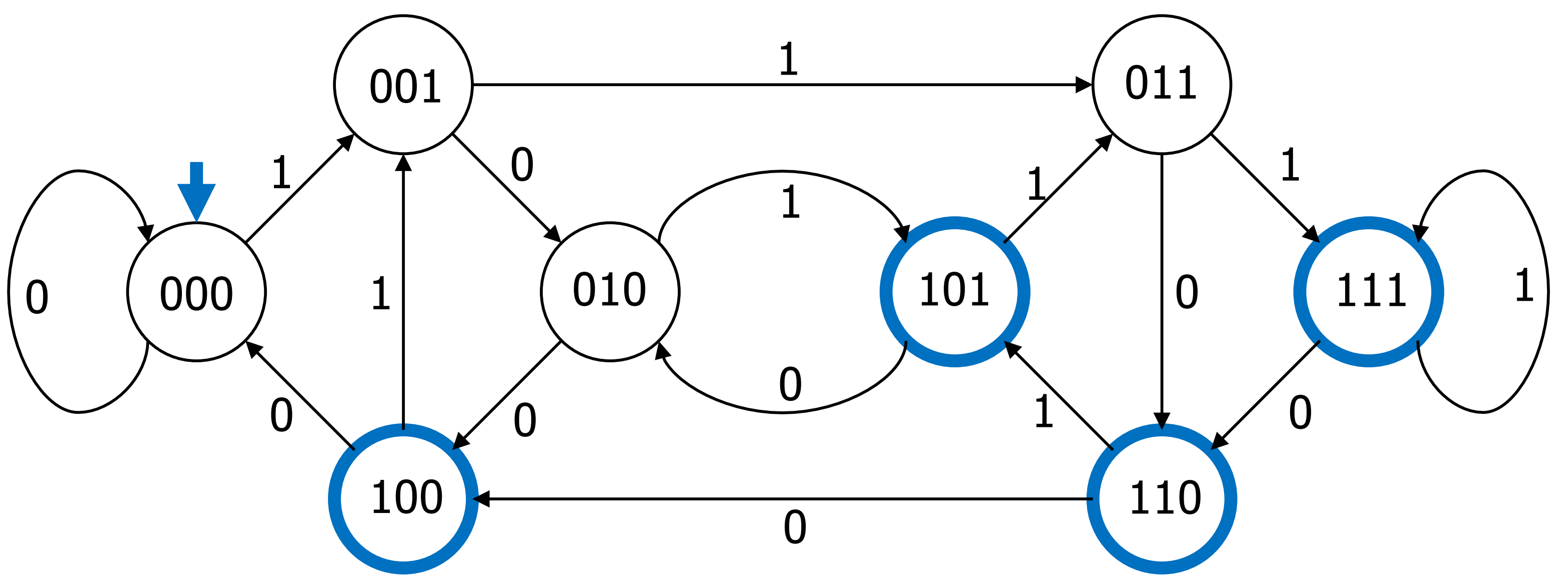
3 bit shift register “Remember the last three bits”



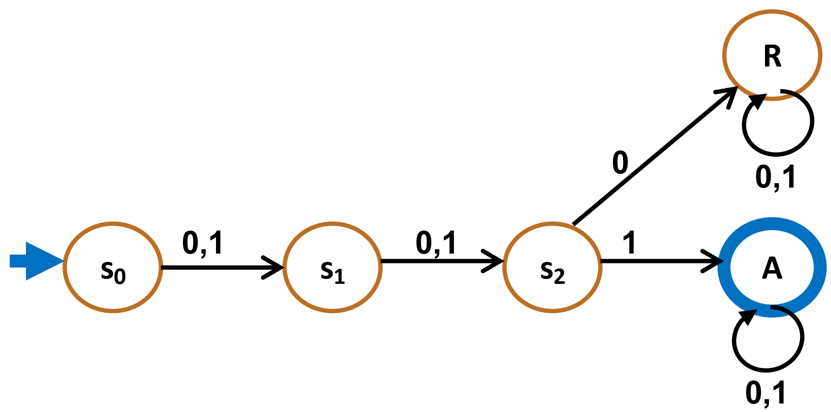
The set of binary strings with a 1 in the 3rd position from the end

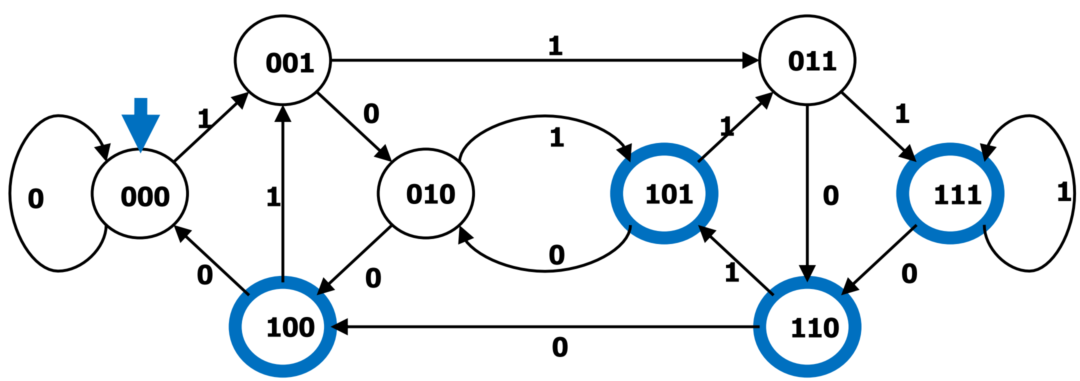


The set of binary strings with a 1 in the 3rd position from the end



The beginning versus the end





Adding Output to Finite State Machines

- So far we have considered finite state machines that just accept/reject strings
  - called “Deterministic Finite Automata” or DFAs

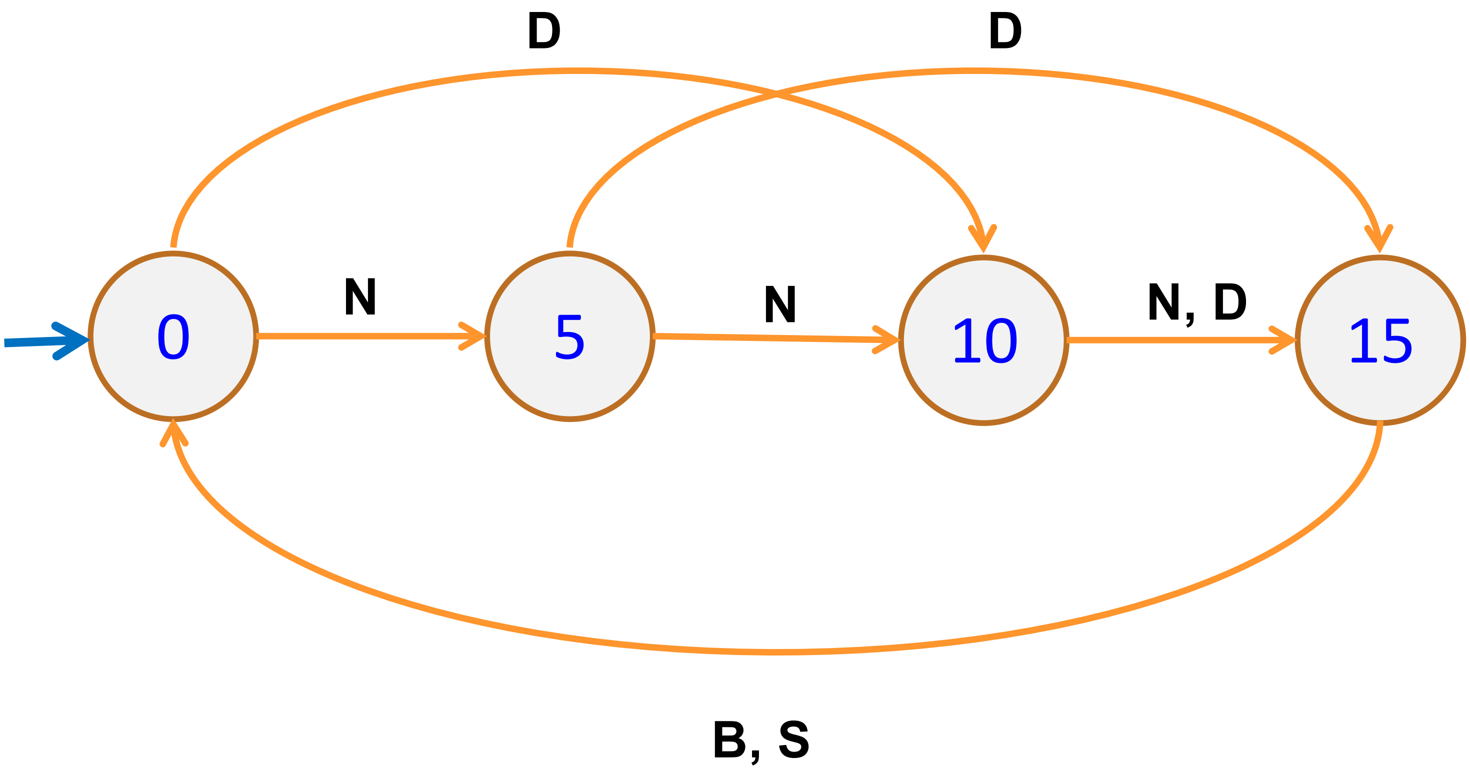
- Now we consider finite state machines *with output*
  - These are the kinds used as controllers

Vending Machine

Enter 15 cents in dimes or nickels

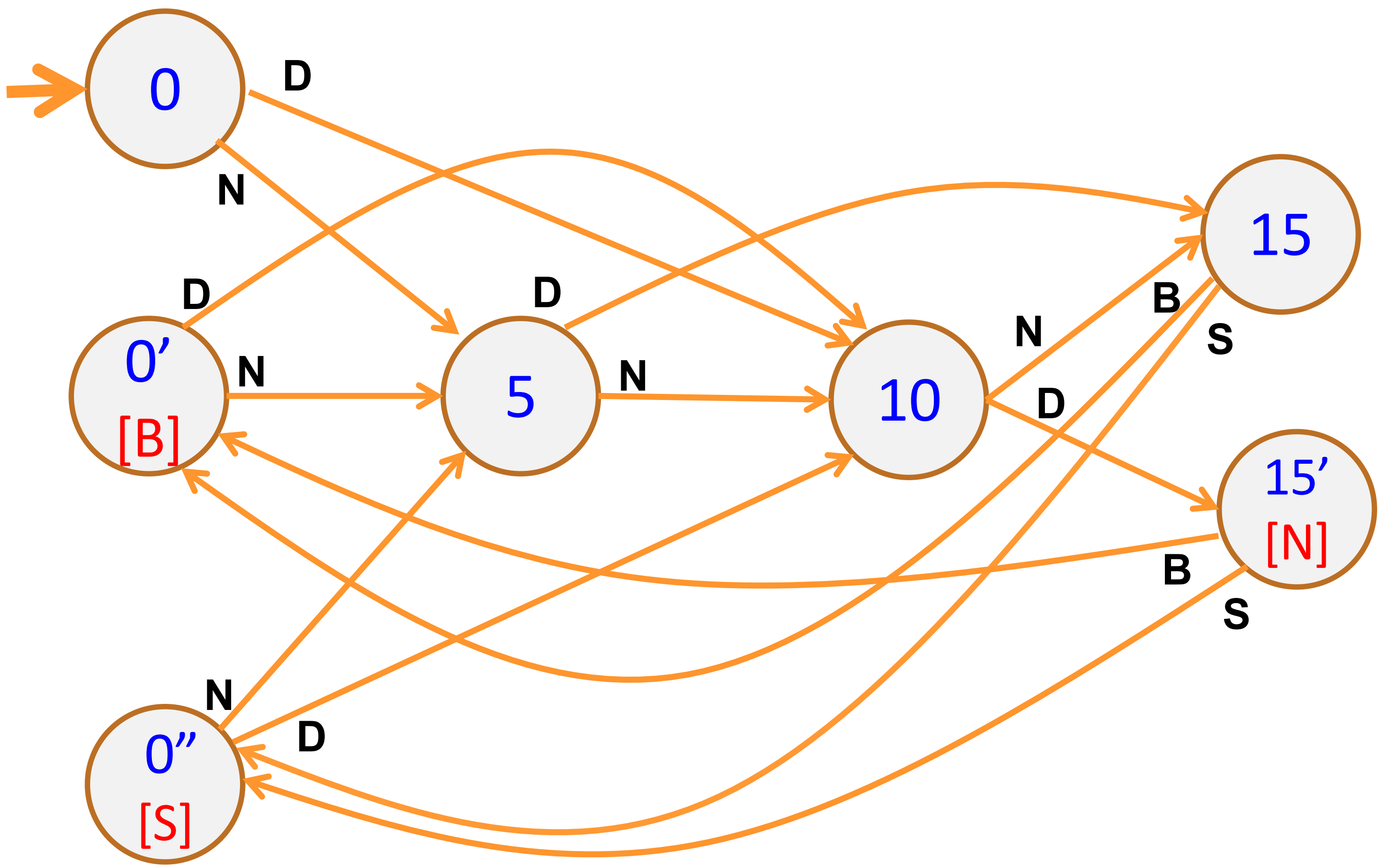
Press S or B for a candy bar

Vending Machine, v0.1



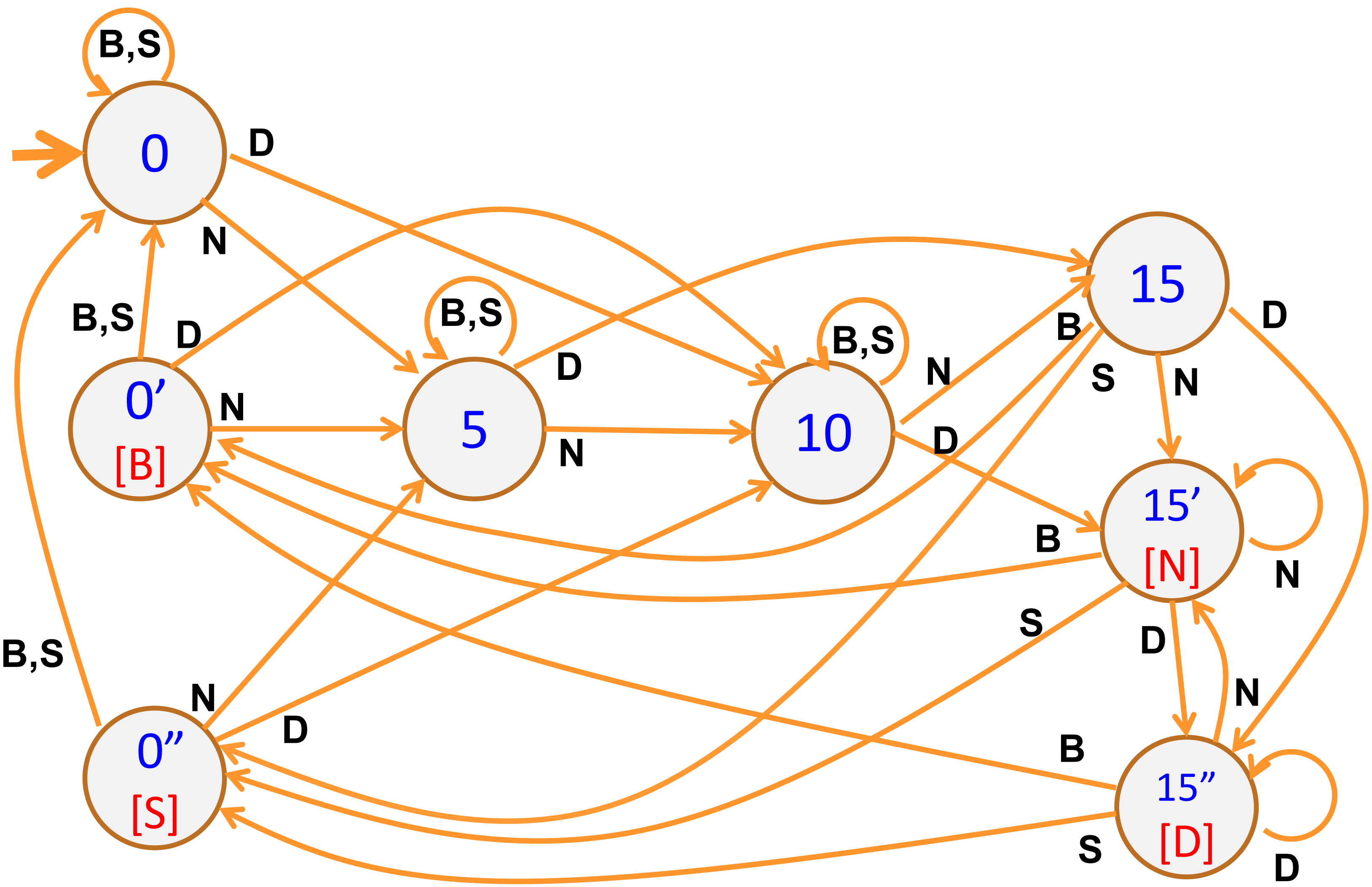
Basic transitions on **N** (nickel), **D** (dime), **B** (butterfinger), **S** (snickers)

Vending Machine, v0.2



Adding output to states: N – Nickel, S – Snickers, B – Butterfinger

Vending Machine, v1.0



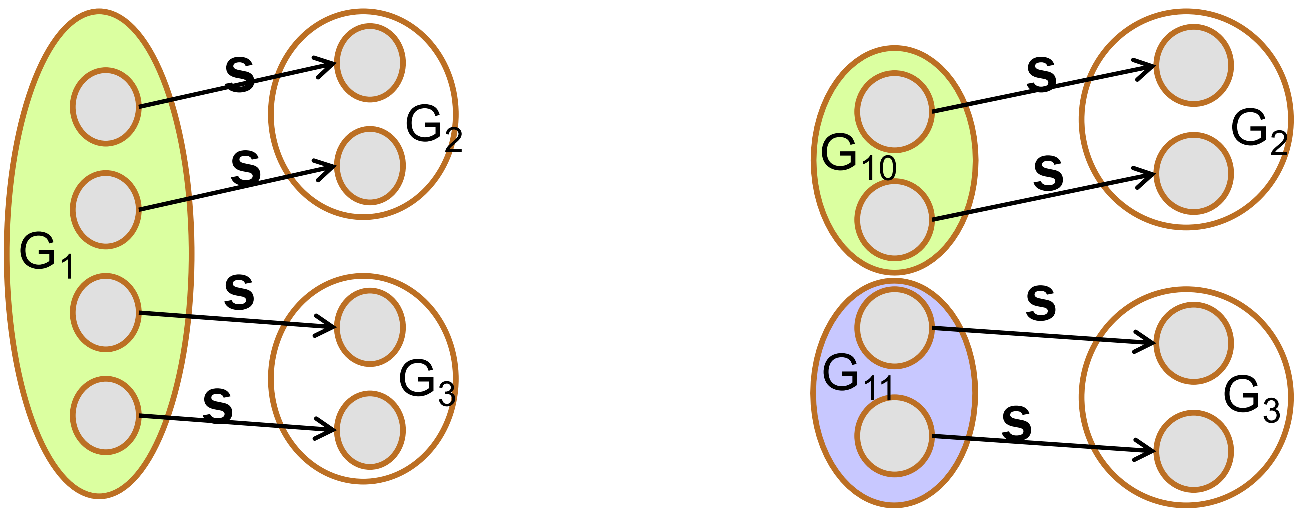
Adding additional “unexpected” transitions to cover all symbols for each state

State Minimization

- Many different FSMs (DFAs) for the same problem
- Take a given FSM and try to reduce its state set by combining states
  - Algorithm will always produce the unique minimal equivalent machine (up to renaming of states) but we won't prove this

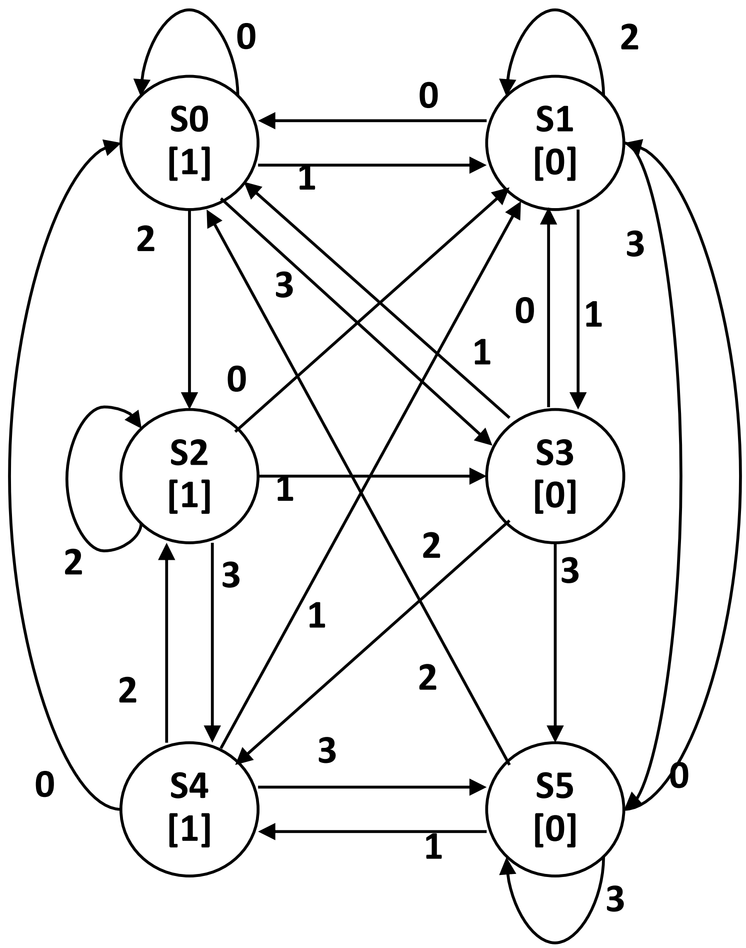
State Minimization Algorithm

1. Put states into groups based on their outputs (or whether they are final states or not)
2. Repeat the following until no change happens
   a. If there is a symbol s so that not all states in a group G agree on which group s leads to, split G into smaller groups based on which group the states go to on s



3. Finally, convert groups to states

State Minimization Example

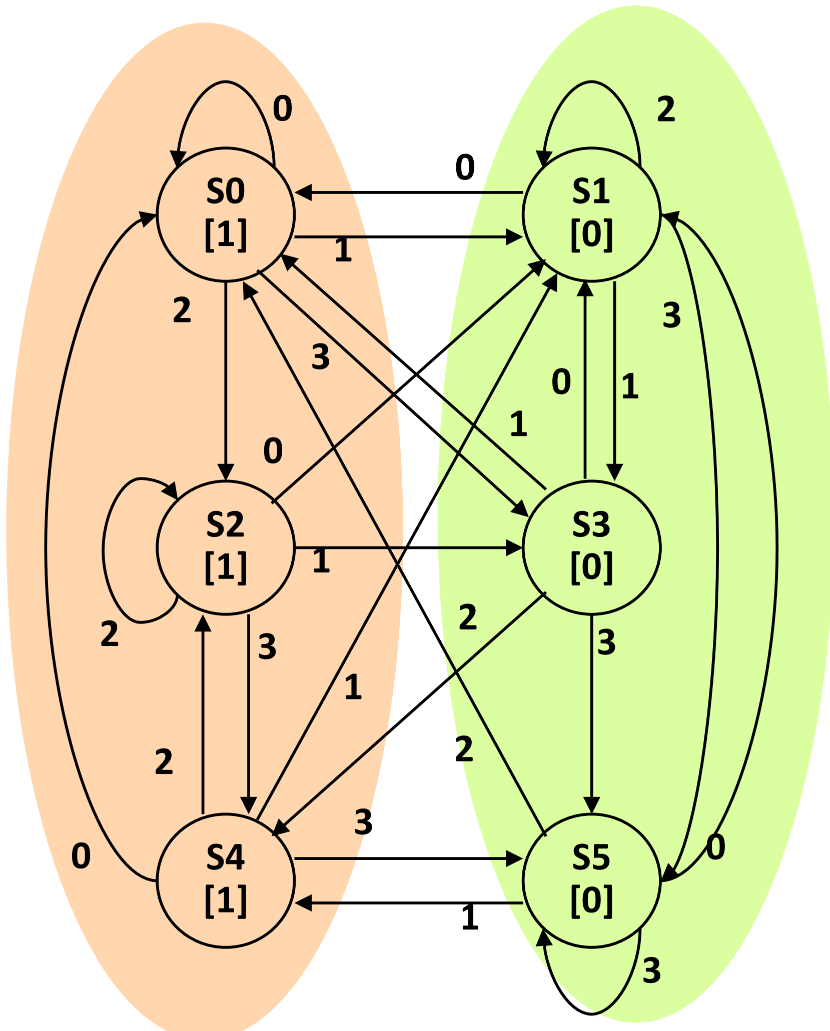


present state	next state 0	1	2	3	output
S0	S0	S1	S2	S3	1
S1	S0	S3	S1	S5	0
S2	S1	S3	S2	S4	1
S3	S1	S0	S4	S5	0
S4	S0	S1	S2	S5	1
S5	S1	S4	S0	S5	0

state transition table

Put states into groups based on their outputs (or whether they are final states or not)

State Minimization Example



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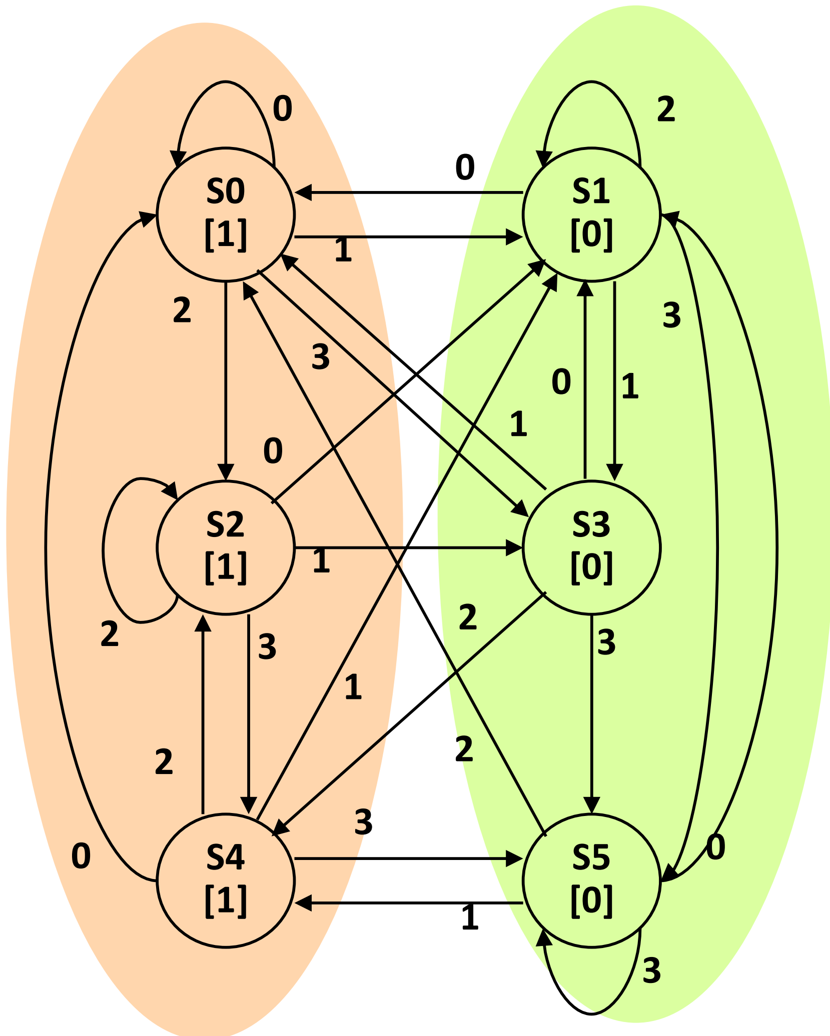
S3	S1	S0	S4	S5	0
S4	S0	S1	S2	S5	1
S5	S1	S4	S0	S5	0

state
transition table

Put states into groups based on their outputs (or whether they are final states or not)

State Minimization Example



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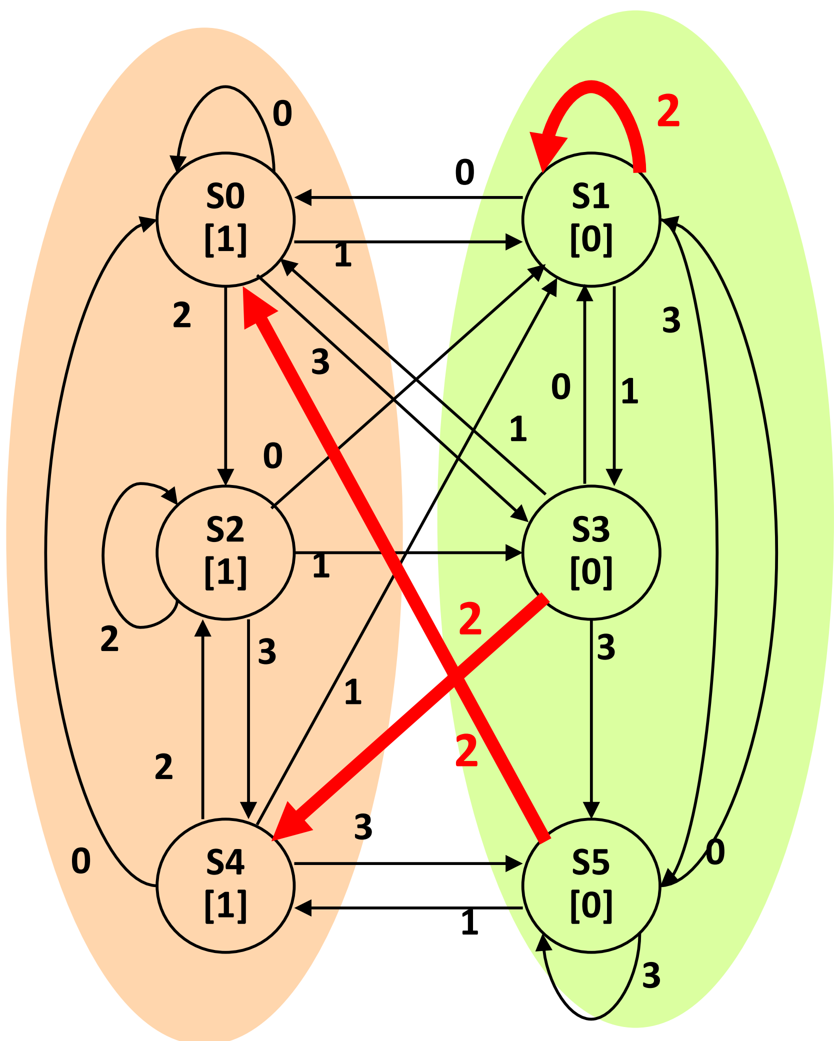
S4	S0	S1	S2	S5	1
S5	S1	S4	S0	S5	0

state transition table

Put states into groups based on their outputs (or whether they are final states or not)

If there is a symbol **s** so that not all states in a group **G** agree on which group **s** leads to, split **G** based on which group the states go to on **s**

State Minimization Example



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state transition table

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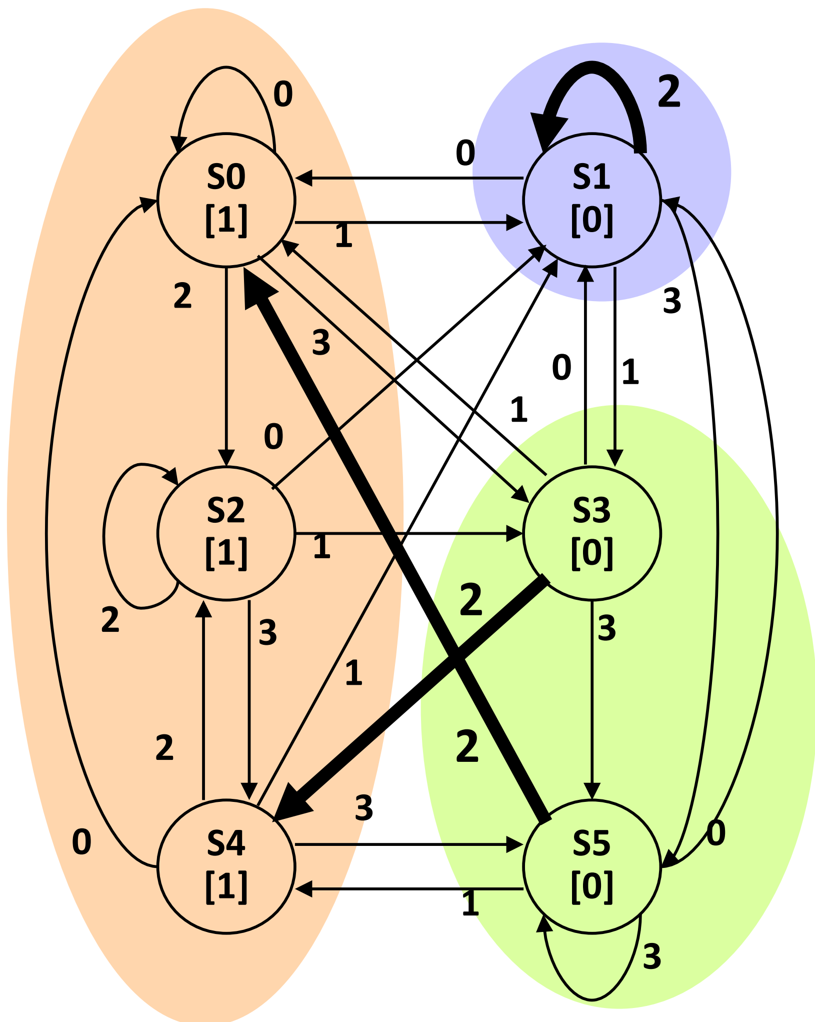

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state transition table

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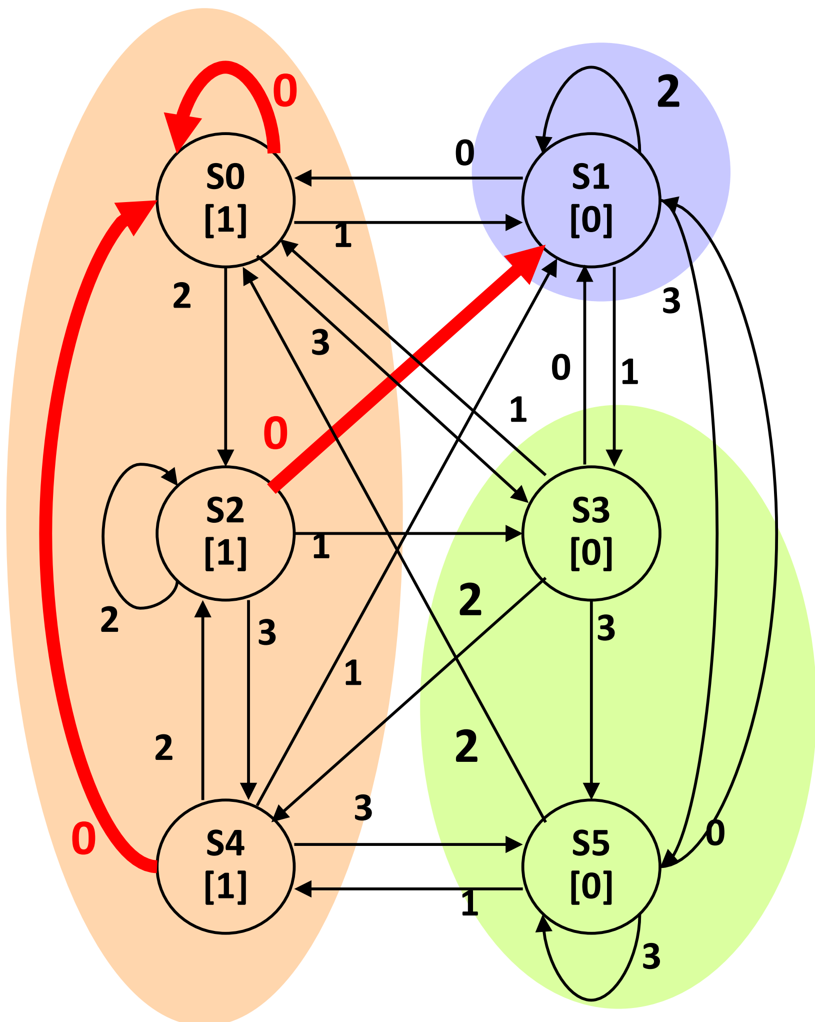

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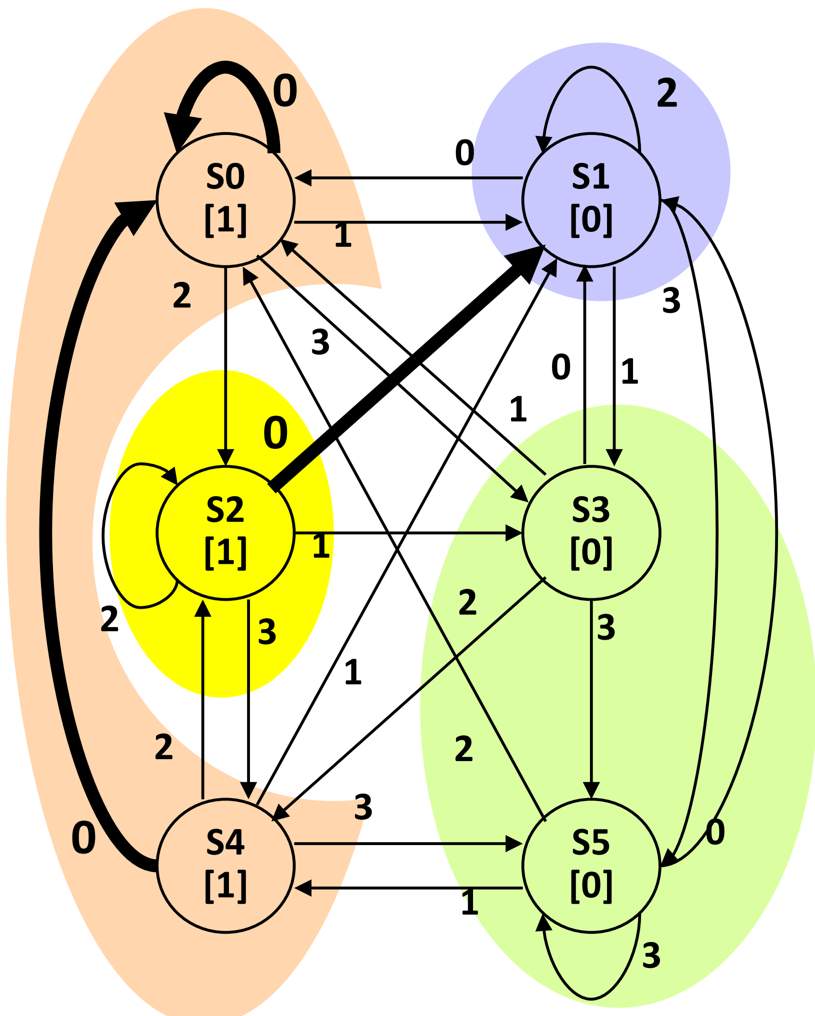
state transition table

Put states into groups based on their outputs (or whether they are final states or not)

If there is a symbol **s** so that not all states in a group **G** agree on which group **s** leads to, split **G** based on which group the states go to on **s**

State Minimization Example



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state transition table

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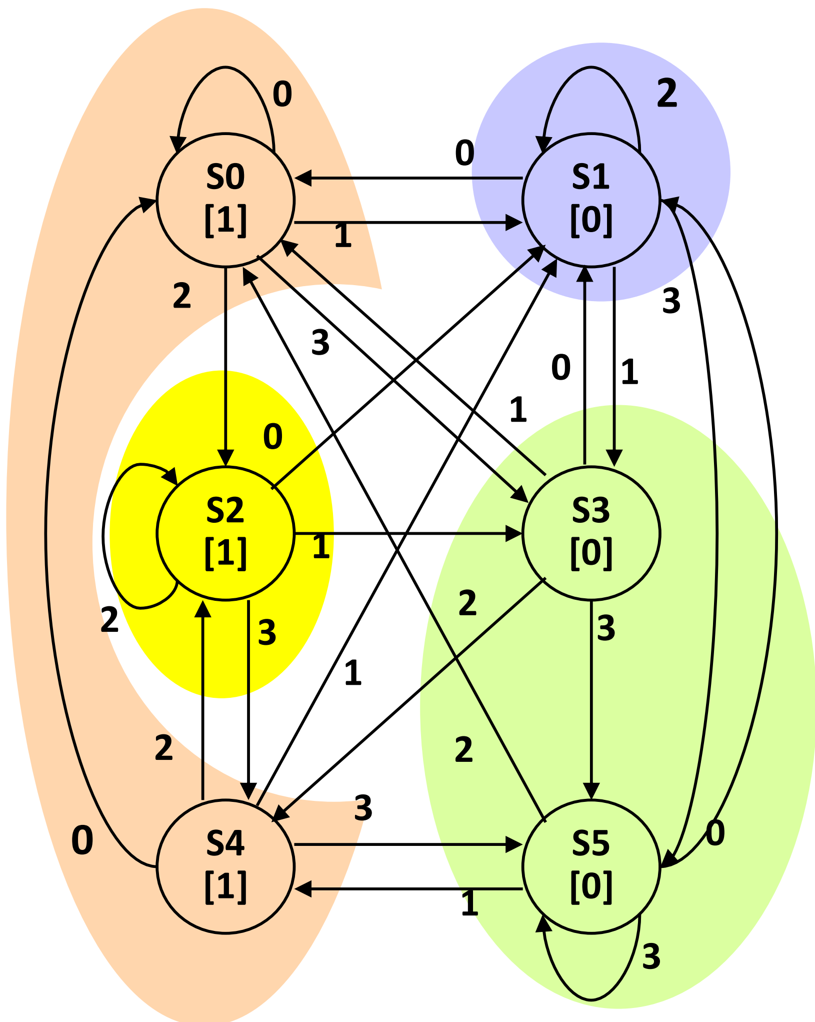
State Minimization Example



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S2	S1	S3	S2	S4	1
S3	S1	S0	S4	S5	0
S4	S0	S1	S2	S5	1
S5	S1	S4	S0	S5	0

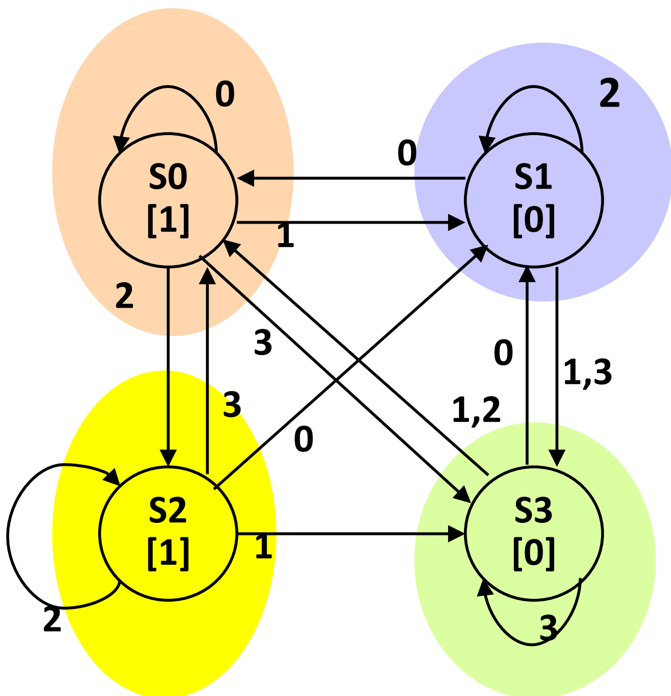
state transition table

Finally convert groups to states:

Can combine states S0-S4 and S3-S5.

In table replace all S4 with S0 and all S5 with S3

Minimized Machine



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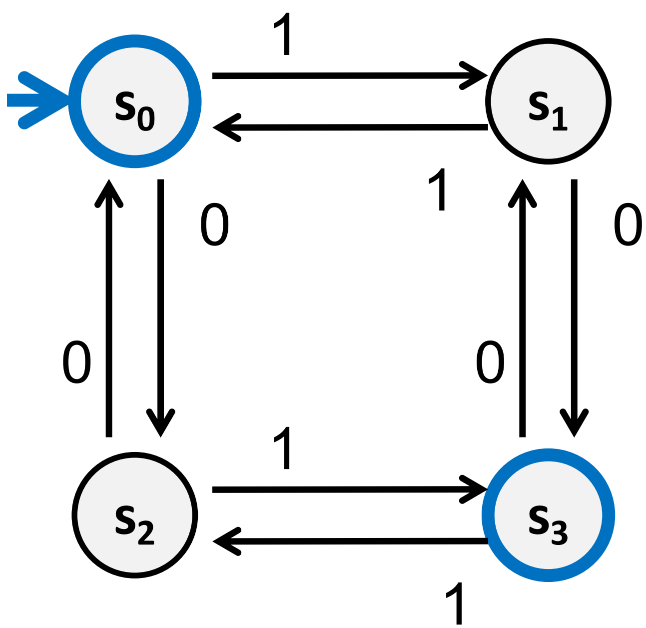
S3	S1	S0	S0	S3	0

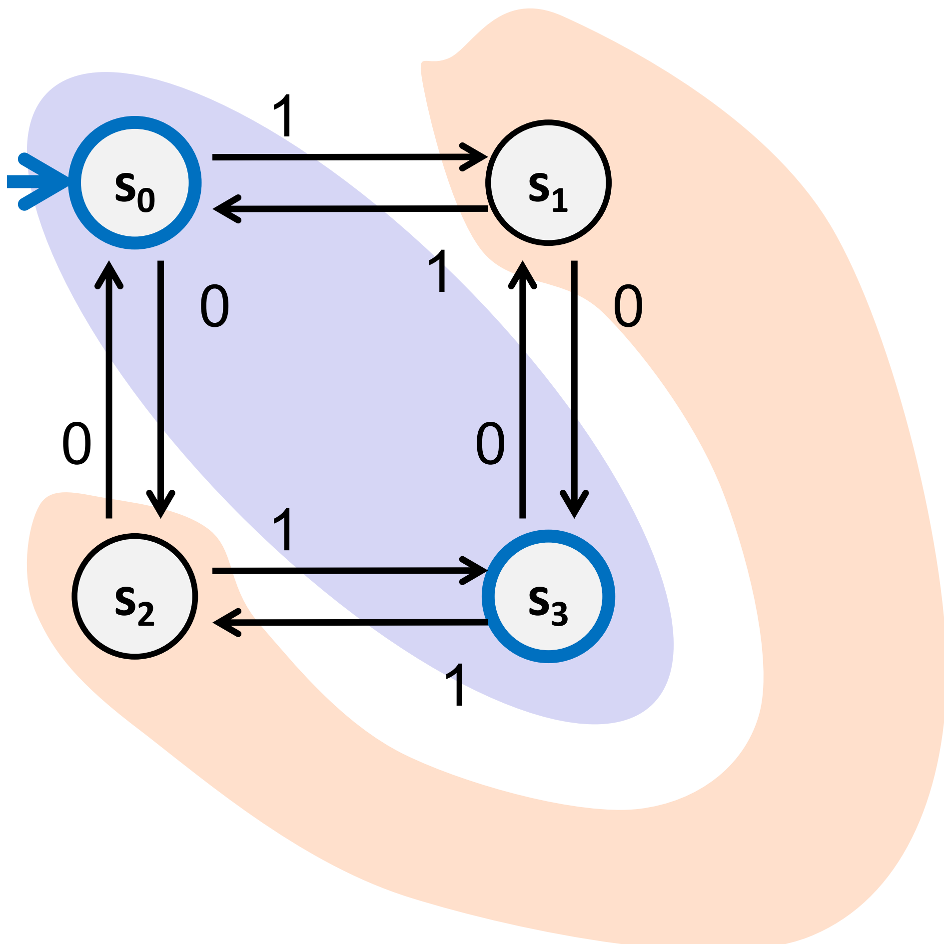
state
transition table

A Simpler Minimization Example



The set of all binary strings with # of 1's ≡ # of 0's (mod 2).

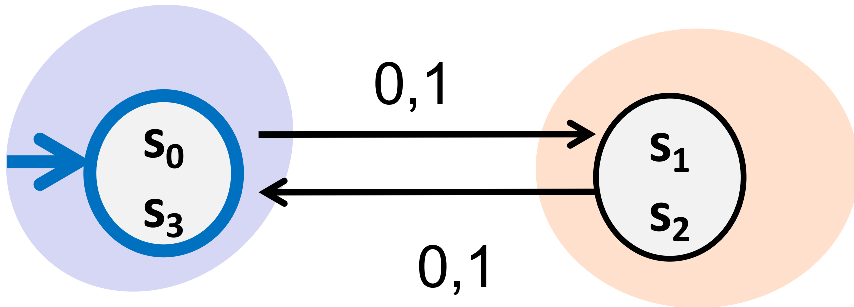
A Simpler Minimization Example



Split states into final/non-final groups

Every symbol causes the DFA to go from one group to the other so neither group needs to be split

Minimized DFA



The set of all binary strings with even length.

Partial Correctness of Minimization Algorithm

- **Prove this claim: after processing input x, if the old machine was in state q, then the new machine is in the state S with $q \in S$**
  - **True after 0 characters processed**
  - **If true after k characters processed, then it's true after k+1 characters processed:**

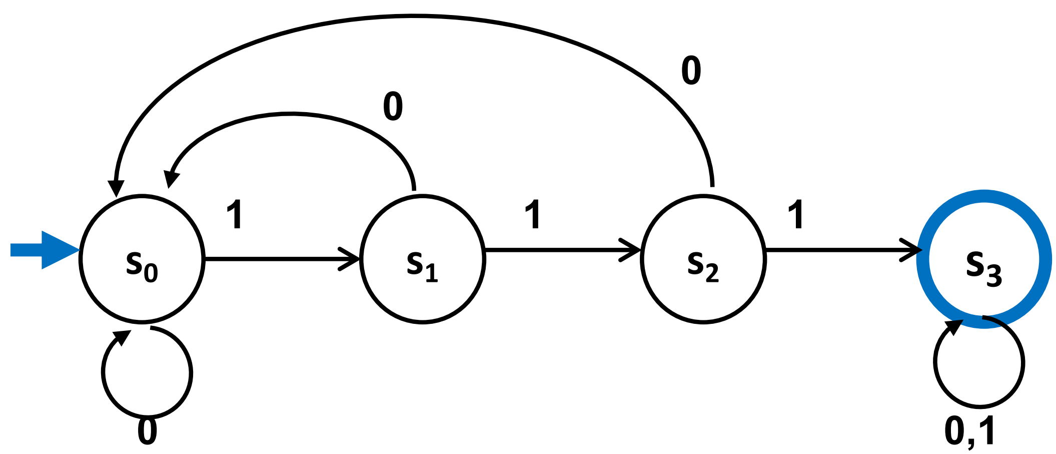
    By inductive hypothesis, after k steps, old machine is in state q and new one in state S with $q \in S$

    By construction, every $r \in S$ is taken to the same state S' on input x_{k+1}, so q is taken to some $q' \in S'$.
- **At end, since every $r \in S$ is accepting or rejecting, new machine gives correct answer.**

Another way to look at DFAs

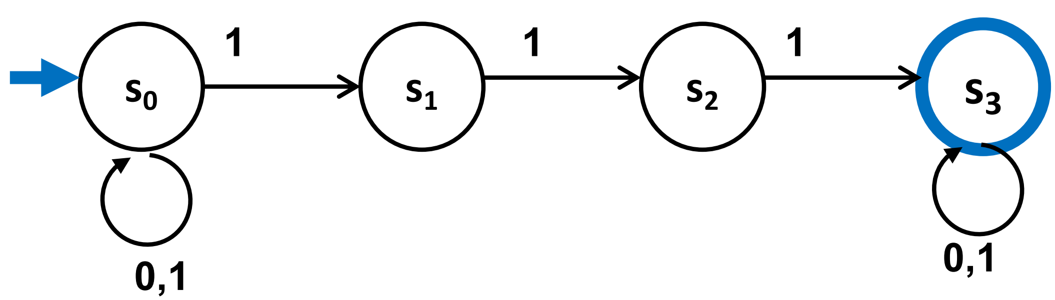
Definition: The label of a path in a DFA is the concatenation of all the labels on its edges in order

Lemma: x is in the language recognized by a DFA iff
x labels a path from the start state to some accepting state

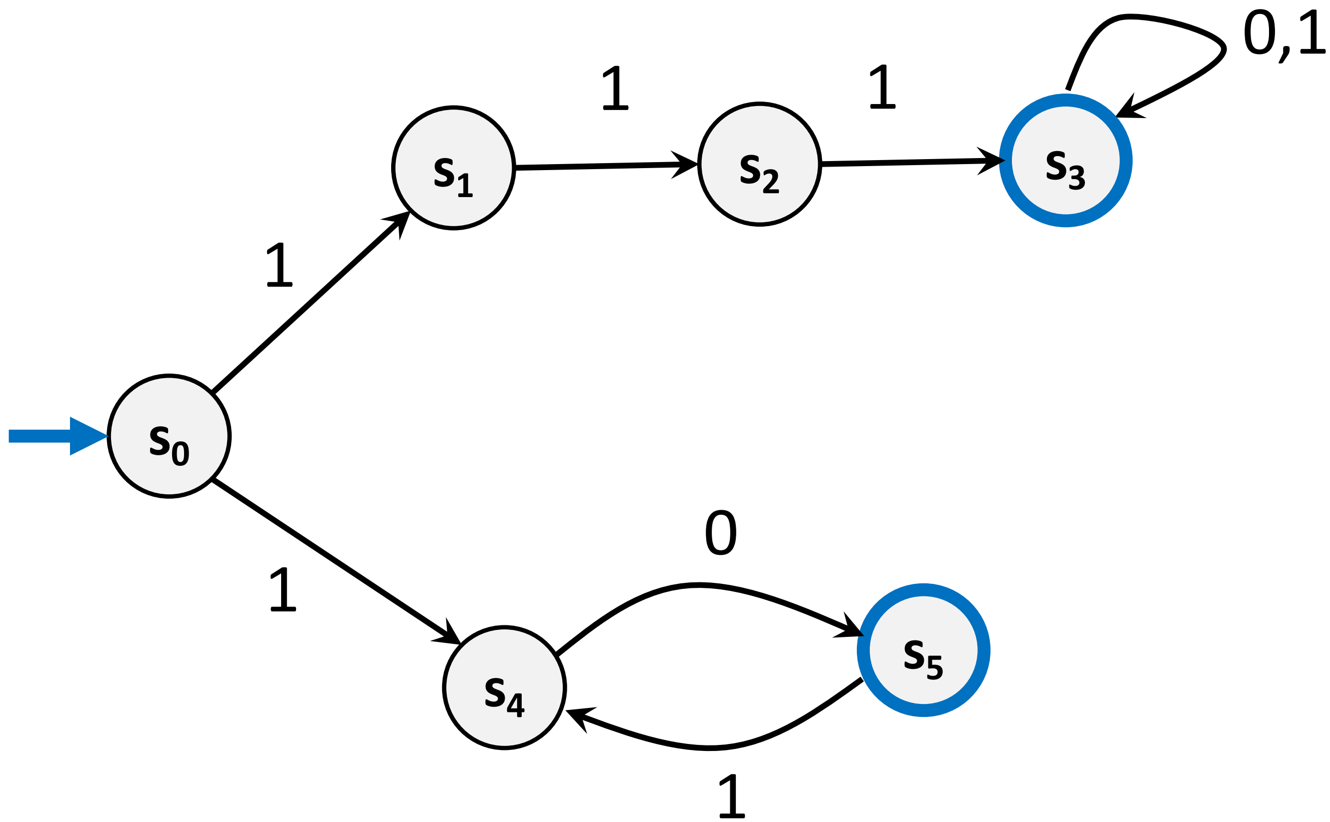


Nondeterministic Finite Automata (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol— can have 0 or >1
  - Also can have edges labeled by empty string ε
- **Definition:** x is in the language recognized by an NFA if and only if x labels some path from the start state to an accepting state

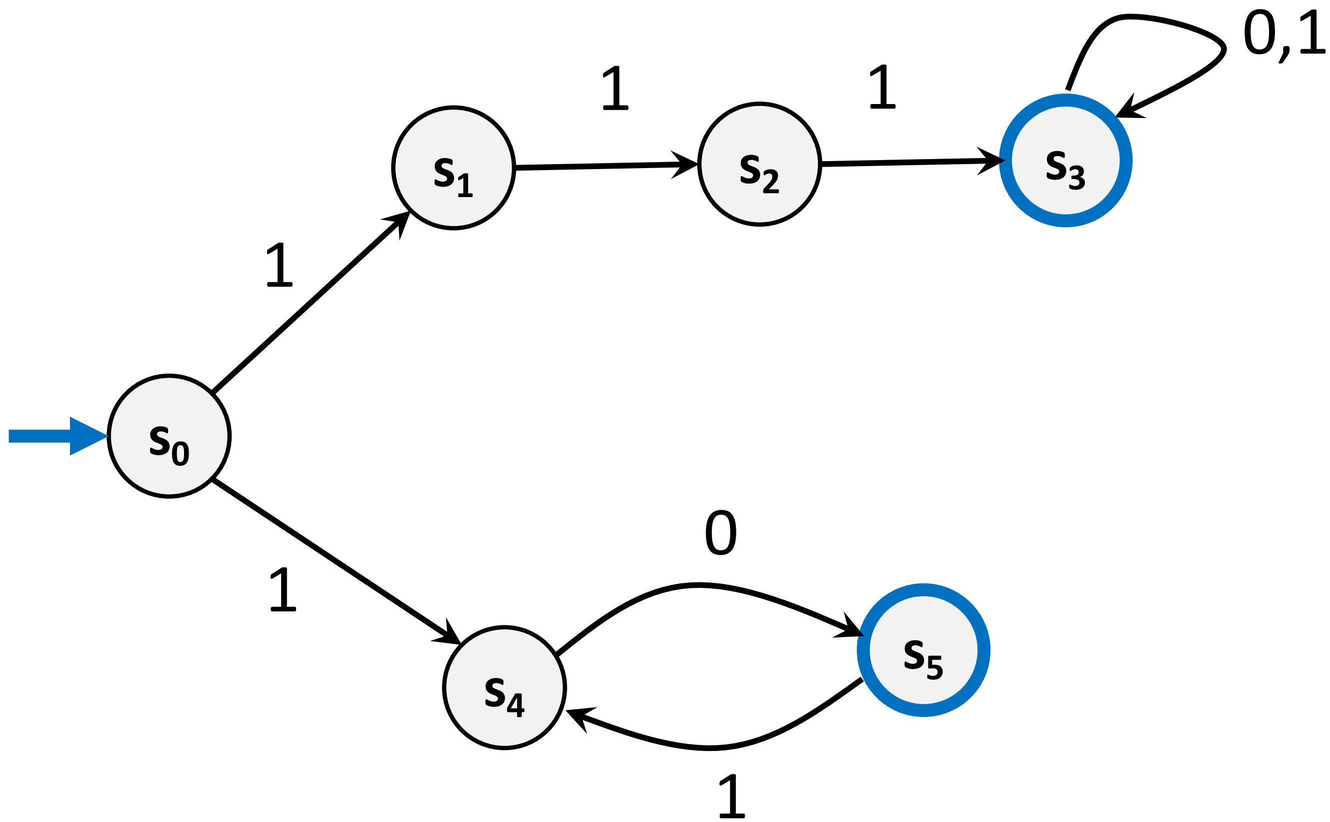


Consider This NFA



What language does this NFA accept?

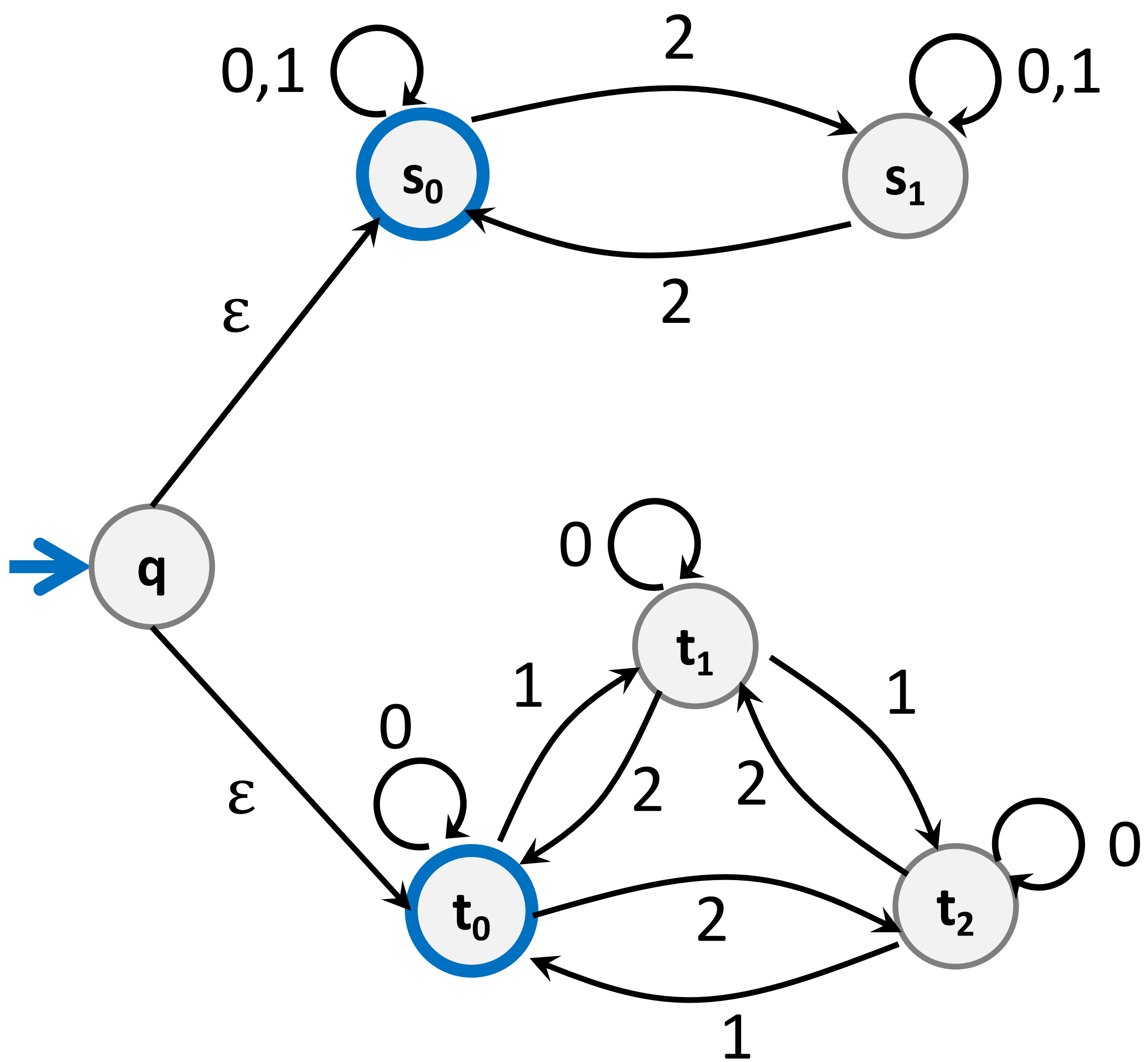
Consider This NFA



What language does this NFA accept?

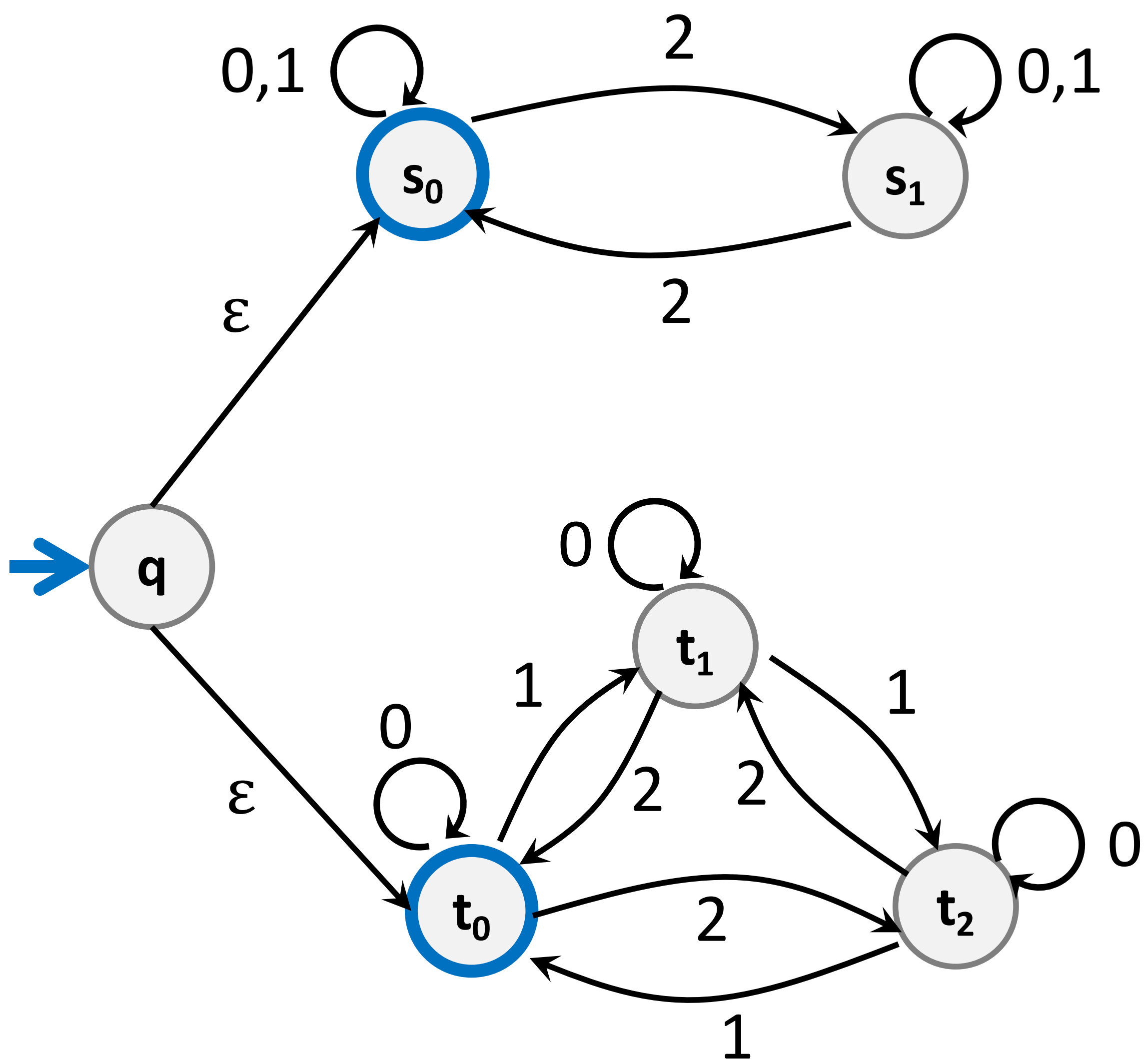
10(10)* U 111 (0 U 1)*

NFA ε-moves



NFA ε-moves

Strings over {0,1,2} w/even # of 2's OR sum to 0 mod 3



Three ways of thinking about NFAs

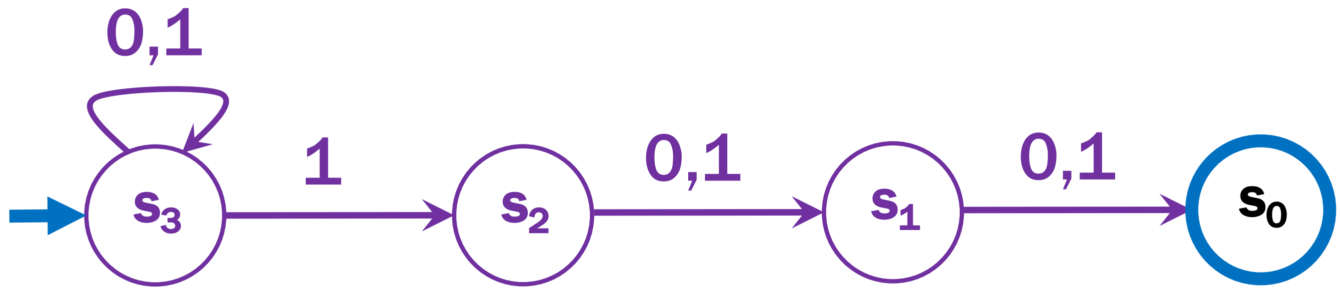
- Outside observer: Is there a path labeled by x from the start state to some accepting state?

- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

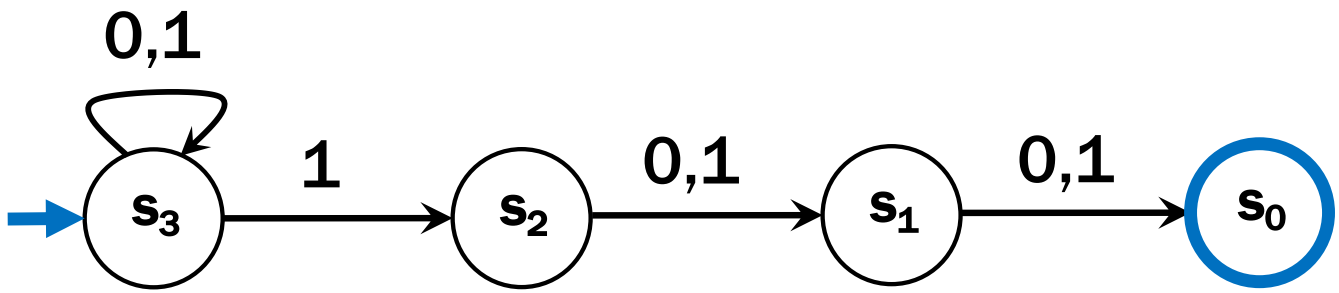
- Perfect guesser: The NFA has input x and whenever there is a choice of what to do it magically guesses a good one (if one exists)

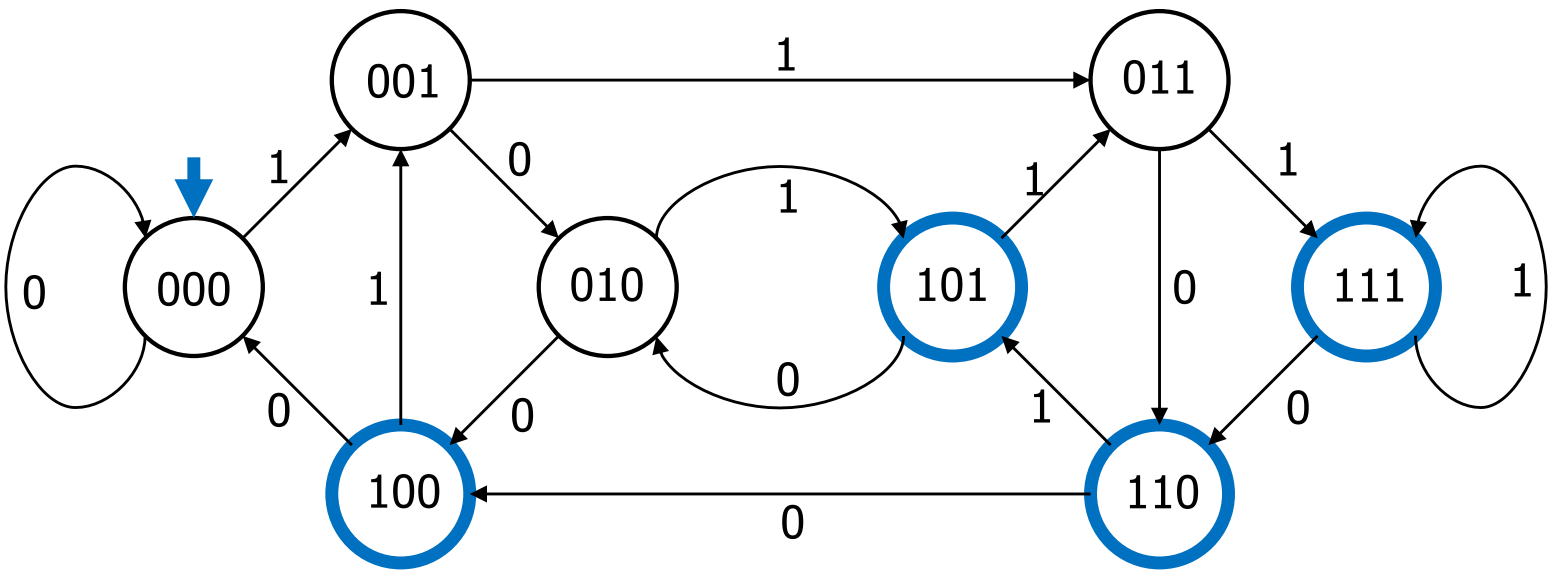
NFA for set of binary strings with a 1 in the 3^{rd} position from the end

NFA for set of binary strings with a 1 in the 3rd position from the end

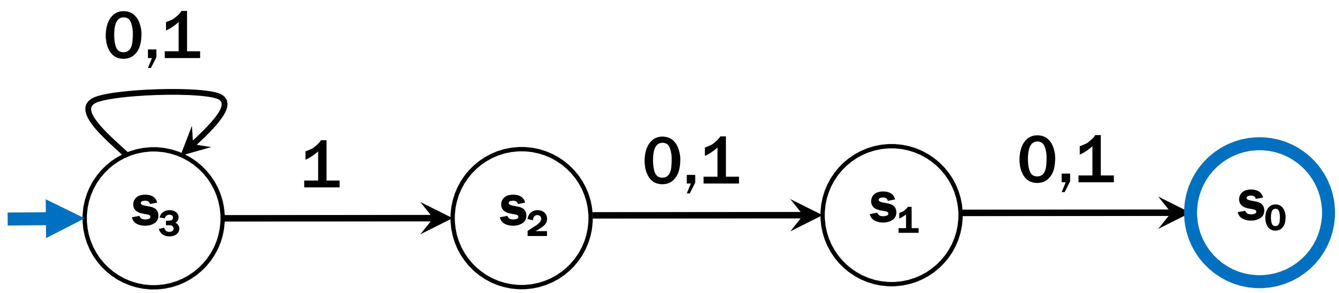


Compare with the smallest DFA





Parallel Exploration view of an NFA



Input string 0101100