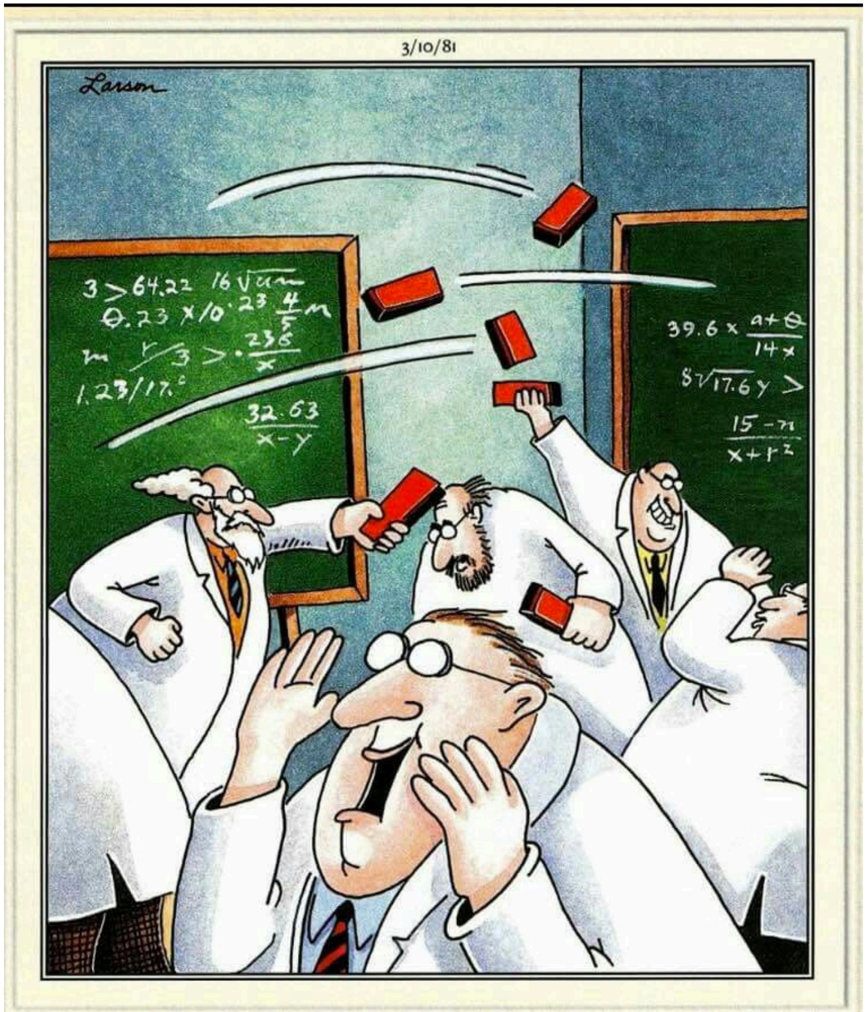
CSE 311: Foundations of Computing

Lecture 7: Logical Inference



“Eraser fight!!”

Last class: Bound vs Free

- **Variables are bound *within an expression* if there is an earlier declaration of the variable**

```
float totalArea(float[] radii) {
  float sum = 0;
  for (int j = 0; j < radii.length; j++) {
    sum += 2 * PI * radii[j];
  }
  return sum;
}
```

Last class: Nested Quantifiers

- **Bound variable names don't matter**

$$\forall x\, \exists y\, P(x, y) \equiv \forall a\, \exists b\, P(a, b)$$

- **Positions of quantifiers can <u>sometimes</u> change**

$$\forall x\, (Q(x) \land \exists y\, P(x, y)) \equiv \forall x\, \exists y\, (Q(x) \land P(x, y))$$

- **But: order is important...**

Last class: Quantifier Order Can Matter

Domain of Discourse
Integers OR {1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= "x ≥ y"

"There is a number greater than or equal to all numbers."

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

Last class: Quantifier Order Can Matter

Domain of Discourse

Integers

OR

{1, 2, 3, 4}

Predicate Definitions

GreaterEq(x, y) ::= "$x \geq y$"

"There is a number greater than or equal to all numbers."

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)$$

"Every number has a number greater than or equal to it."

$$\forall y\ \exists x\ \text{GreaterEq}(x, y)$$

y

x	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

Quantifier Order Can Matter

Domain of Discourse
Integers OR {1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= "x ≥ y"

“There is a number greater than or equal to all numbers.”

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)$$

“Every number has a number greater than or equal to it.”

$$\forall y\ \exists x\ \text{GreaterEq}(x, y)$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Important: both include the case x = y

Different names does not imply different objects!

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other
- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

New Perspective

Rather than comparing P and Q as columns, zooming in on just the rows where P is true:

p	*q*	P	Q
T	T	T	
T	F	T	
F	T	F	
F	F	F	

New Perspective

Rather than comparing **P** and **Q** as columns, zooming in on just the rows where P is true:

p	*q*	**P**	**Q**
T	T	T	T
T	F	T	T
F	T	F	
F	F	F	

Given that P is true, we see that Q is also true.

$$P \Rightarrow Q$$

New Perspective

Rather than comparing **P** and **Q** as columns, zooming in on just the rows where P is true:

p	*q*	P	Q
T	T	T	T
T	F	T	T
F	T	F	T
F	F	F	F

When we step back, what have we proven?

New Perspective

Rather than comparing **P** and **Q** as columns, zooming in on just the rows where **P** is true:

p	*q*	**P**	**Q**	**P → Q**
T	T	T	T	T
T	F	T	T	T
F	T	F	T	T
F	F	F	F	T

When we step back, what have we proven?

$$(P \rightarrow Q) \equiv \mathbf{T}$$

New Perspective

Equivalences

$P \equiv Q$ and $(P \leftrightarrow Q) \equiv T$ are the same

Inference

$P \Rightarrow Q$ and $(P \rightarrow Q) \equiv T$ are the same

Can do the inference by zooming in
to the rows where P is true

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with given facts (hypotheses)
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If A and $A \rightarrow B$ are both true, then B must be true
- Write this rule as $$\frac{A \,;\, A \rightarrow B}{\therefore\ B}$$
- Given:
  - If it is Wednesday, then you have a 311 class today.
  - It is Wednesday.
- Therefore, by Modus Ponens:
  - You have a 311 class today.

My First Proof!

Show that r follows from p, $p \to q$, and $q \to r$

1.	p	Given
2.	$p \to q$	Given
3.	$q \to r$	Given
4.		
5.		

Modus Ponens $\dfrac{A\ ;\ A \to B}{\therefore\ B}$

My First Proof!

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.	q	MP: 1, 2
5.	r	MP: 3, 4

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

Proofs can use equivalences too

Show that $\neg p$ follows from $p \to q$ and $\neg q$

1.	$p \to q$	Given
2.	$\neg q$	Given
3.	$\boxed{\neg q \to \neg p}$	Contrapositive: 1
4.	$\neg p$	MP: 2, 3

Modus Ponens $\dfrac{A\ ;\ A \to B}{\therefore\ B}$

Inference Rules

If A is true and B is true ....

Requirements: $\underline{\;A \;\; ; \;\; B\;}$

Conclusions: $\therefore C \;\; , \;\; D$

Then, C must be true

Then D must be true

Example (Modus Ponens):

$$\frac{A \;\; ; \;\; A \rightarrow B}{\therefore \qquad B}$$

If I have A and A $\rightarrow$ B both true,
Then B must be true.

Axioms: Special inference rules

If I have nothing...

Requirements: $\overline{}$

Conclusions: $\therefore C \; , \; D$

Then, C must be true

Then D must be true

Example (Excluded Middle):

$$\frac{}{\therefore A \vee \neg A}$$

A $\vee \neg$A must be true.

Simple Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \land B}{\therefore A, B}$$

Intro ∧

$$\frac{A \,;\, B}{\therefore A \land B}$$

Elim ∨

$$\frac{A \lor B \,;\, \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \lor B, B \lor A}$$

Modus Ponens

$$\frac{A \,;\, A \rightarrow B}{\therefore B}$$

Direct Proof Rule

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules

Proofs

Show that r follows from p, $p \to q$ and $(p \wedge q) \to r$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{A\ ;\ A \to B}{\therefore\ B}$$

$$\frac{A \wedge B}{\therefore\ A,\ B}$$

$$\frac{A\ ;\ B}{\therefore\ A \wedge B}$$

Proofs

Show that r follows from $p, p \to q,$ and $p \land q \to r$

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!

1.	p	Given
2.	$p \to q$	Given
3.	q	MP: 1, 2
4.	$p \land q$	Intro $\land$: 1, 3
5.	$p \land q \to r$	Given
6.	r	MP: 4, 5

$$\dfrac{\dfrac{p \quad ; \quad \dfrac{p \quad ; \quad p \to q}{q}\,\text{MP}}{p \land q}\,\text{Intro } \land \quad ; \quad p \land q \to r}{r}\,\text{MP}$$

Proofs

Prove that $\neg r$ follows from $p \land s$, $q \rightarrow \neg r$, and $\neg s \lor q$.

1.	$p \land s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \lor q$	Given
20.	$\neg r$	?

<u>**First: Write down givens and goal**</u>

<u>**Idea: Work backwards!**</u>

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- We can use $q \rightarrow \neg r$ to get there.
- The justification between 2 and 20 looks like "elim $\rightarrow$" which is MP.

20.	$\neg r$	MP: 2, ?

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- Now, we have a new "hole"
- We need to prove q...
  - Notice that at this point, if we prove q, we've proven $\neg r$...

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \to \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \to \neg r$	Given
3.	$\neg s \vee q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

This looks like or-elimination.

$$\text{Elim} \vee \quad \frac{A \vee B \,;\, \neg A}{\therefore B}$$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
18.	$\neg\neg s$	?
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

$\neg\neg s$ doesn't show up in the givens but s does and we can use equivalences

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
17.	s	?
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
17.	s	$\wedge$ Elim: 1
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

No holes left! We just need to clean up a bit.

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given
4. s $\wedge$ Elim: 1
5. $\neg\neg s$ Double Negation: 4
6. q $\vee$ Elim: 3, 5
7. $\neg r$ MP: 2, 6

Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \rightarrow r$ given

2. $\cancel{(p \vee q) \rightarrow r}$ ~~intro~~ $\cancel{\vee}$ ~~from 1.~~

Does not follow! e.g . p=F, q=T, r=F

To Prove An Implication: $A \rightarrow B$

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

- We use the direct proof rule
- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is **a proof** that "**Given** A, we can prove B."
- The direct proof rule:

  If you have such a proof then you can conclude that $A \rightarrow B$ is true

To Prove An Implication: $A \rightarrow B$

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- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is **a proof** that "**Given** A, we can prove B**.**"
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  If you have such a proof then you can conclude that $A \rightarrow B$ is true

Example: Prove $p \rightarrow (p \vee q)$.

proof subroutine

Indent proof subroutine $\Rightarrow$

1.1.	p	Assumption
1.2.	$p \vee q$	Intro $\vee$: 1
1.	$p \rightarrow (p \vee q)$	Direct Proof Rule

Proofs using the direct proof rule

Show that $p \to r$ follows from q and $(p \land q) \to r$

1. q Given
2. $(p \land q) \to r$ Given

This is a proof of $p \to r$

3.1. p Assumption

3.2. $p \land q$ Intro $\land$: 1, 3.1

3.3. r MP: 2, 3.2

If we know p is true… Then, we've shown r is true

3. $p \to r$ Direct Proof Rule

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens...

Example

Prove: $(p \land q) \rightarrow (p \lor q)$

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

1.1.	$p \wedge q$	Assumption
1.2.	p	Elim $\wedge$: 1.1
1.3.	$p \vee q$	Intro $\vee$: 1.2
1.	$(p \wedge q) \rightarrow (p \vee q)$	Direct Proof Rule

One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.