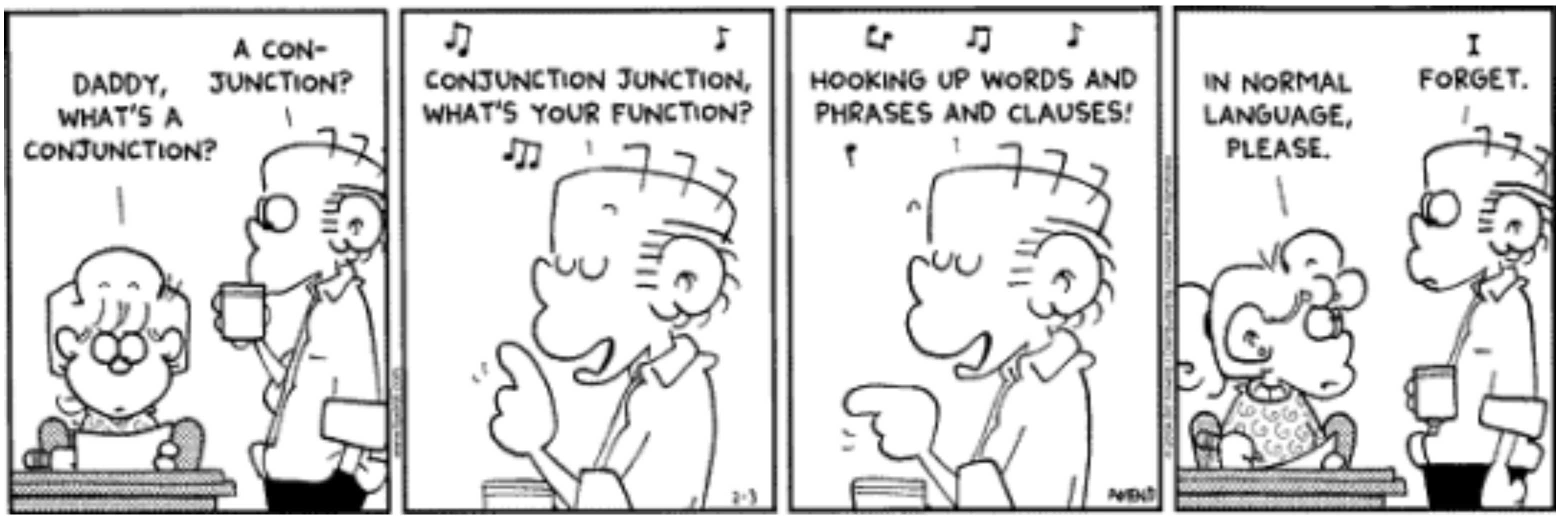
CSE 311: Foundations of Computing

Lecture 2: More Logic, Equivalence & Digital Circuits



Administrivia Updates

- Homework assignments will be due Fridays
  – please start early
- Typesetting information on web site
  – not required but ensures legible assignments

Homework #1

- **Posted today. Due next Friday**
  - includes some material from Monday
  - some problems require experimentation

- **Submit your solution via GradeScope**
  - You should receive an email invitation by Monday evening
  - If you don't receive one, send e-mail to cse311-staff@cs.washington.edu

Last class: Some Connectives & Truth Tables

Negation (not)

p	$\neg p$
T	F
F	T

Conjunction (and)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (or)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Last class: Implication

"If it's raining, then I have my umbrella"

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In English, we can also write

"I have my umbrella if it's raining."

$p \rightarrow q$

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(1) "I have collected all 151 Pokémon if I am a Pokémon master"

(2) "I have collected all 151 Pokémon only if I am a Pokémon master"

$p \rightarrow q$

(1) "I have collected all 151 Pokémon if I am a Pokémon master"

(2) "I have collected all 151 Pokémon only if I am a Pokémon master"

These sentences are implications in opposite directions:

(1) "Pokémon masters have all 151 Pokémon"

(2) "People who have 151 Pokémon are Pokémon masters"

So, the implications are:

(1) If I am a Pokémon master, *then* I have collected all 151 Pokémon.

(2) If I have collected all 151 Pokémon, *then* I am a Pokémon master.

$p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication:

- *p* implies *q*
- whenever *p* is true *q* must be true
- if *p* then *q*
- *q* if *p*
- *p* is sufficient for *q*
- *p* only if *q*
- *q is necessary for p*

Biconditional: $p \leftrightarrow q$

- *p* iff *q*
- *p* implies *q* and *q* implies *p*
- *p* is necessary and sufficient for *q*

p	q	$p \leftrightarrow q$

Biconditional: $p \leftrightarrow q$

- *p* iff *q*
- *p* implies *q* and *q* implies *p*
- *p* is necessary and sufficient for *q*

p	q	$p \leftrightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Back to Garfield…

p	"Garfield has black stripes"
q	"Garfield is an orange cat"
r	"Garfield likes lasagna"

"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"

↓

(p if (q and r)) and (q or (not r))

↓

(p "if" $(q \wedge r)) \wedge$ **(q** $\vee \neg$**r)**

Back to Garfield…

p	"Garfield has black stripes"
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"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"

↓

(p if (q and r)) and (q or (not r))

↓

$(\mathbf{p}$ **"if"** $(q \wedge r)) \wedge (\mathbf{q} \vee \neg \mathbf{r})$

↓

$((q \wedge r) \longrightarrow \mathbf{p}) \wedge (\mathbf{q} \vee \neg \mathbf{r})$

Analyzing the Garfield Sentence with a Truth Table

p	q	r	$\neg r$	$q \vee \neg r$	$q \wedge r$	$(q \wedge r) \to p$	$((q \wedge r) \to p) \wedge (q \vee \neg r)$
F	F	F					
F	F	T					
F	T	F					
F	T	T					
T	F	F					
T	F	T					
T	T	F					
T	T	T					

Analyzing the Garfield Sentence with a Truth Table

p	q	r	$\neg r$	$q \vee \neg r$	$q \wedge r$	$(q \wedge r) \rightarrow p$	$((q \wedge r) \rightarrow p) \wedge (q \vee \neg r)$
F	F	F	T	T	F	T	T
F	F	T	F	F	F	T	F
F	T	F	T	T	F	T	T
F	T	T	F	T	T	F	F
T	F	F	T	T	F	T	T
T	F	T	F	F	F	T	F
T	T	F	T	T	F	T	T
T	T	T	F	T	T	T	T

Converse, Contrapositive

Implication:

$p \rightarrow q$

Converse:

$q \rightarrow p$

Contrapositive:

$\neg q \rightarrow \neg p$

Inverse:

$\neg p \rightarrow \neg q$

Consider

p: *x* is divisible by 2

q: *x* is divisible by 4

$p \rightarrow q$	
$q \rightarrow p$	
$\neg q \rightarrow \neg p$	
$\neg p \rightarrow \neg q$	

Converse, Contrapositive

Implication:

$$p \to q$$

Converse:

$$q \to p$$

Contrapositive:

$$\neg q \to \neg p$$

Inverse:

$$\neg p \to \neg q$$

Consider

p: x is divisible by 2

q: x is divisible by 4

$p \to q$	
$q \to p$	
$\neg q \to \neg p$	
$\neg p \to \neg q$	

	Divisible By 2	Not Divisible By 2
Divisible By 4		
Not Divisible By 4		

Converse, Contrapositive

Implication:

$p \rightarrow q$

Converse:

$q \rightarrow p$

Contrapositive:

$\neg q \rightarrow \neg p$

Inverse:

$\neg p \rightarrow \neg q$

Consider

p: *x* is divisible by 2

q: *x* is divisible by 4

$p \rightarrow q$	
$q \rightarrow p$	
$\neg q \rightarrow \neg p$	
$\neg p \rightarrow \neg q$	

	Divisible By 2	Not Divisible By 2
Divisible By 4	4,8,12,...	Impossible
Not Divisible By 4	2,6,10,...	1,3,5,...

Converse, Contrapositive

Implication:

$p \rightarrow q$

Contrapositive:

$\neg q \rightarrow \neg p$

Converse:

$q \rightarrow p$

Inverse:

$\neg p \rightarrow \neg q$

How do these relate to each other?

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T						
T	F						
F	T						
F	F						

Converse, Contrapositive

Implication:

$p \to q$

Converse:

$q \to p$

Contrapositive:

$\neg q \to \neg p$

Inverse:

$\neg p \to \neg q$

An implication and it's contrapositive have the same truth value!

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Application: Digital Circuits

Computing With Logic

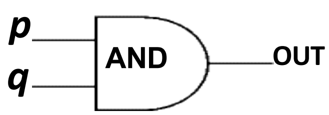
- T corresponds to 1 or “high” voltage
- F corresponds to 0 or “low” voltage

Gates

- Take inputs and produce outputs (functions)
- Several kinds of gates
- Correspond to propositional connectives (most of them)

Last class: AND, OR, NOT Gates

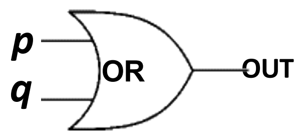
AND Gate



p	*q*	OUT
1	1	1
1	0	0
0	1	0
0	0	0

p	*q*	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

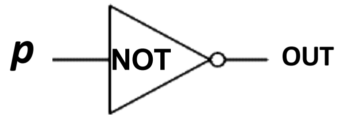
OR Gate



p	*q*	OUT
1	1	1
1	0	1
0	1	1
0	0	0

p	*q*	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

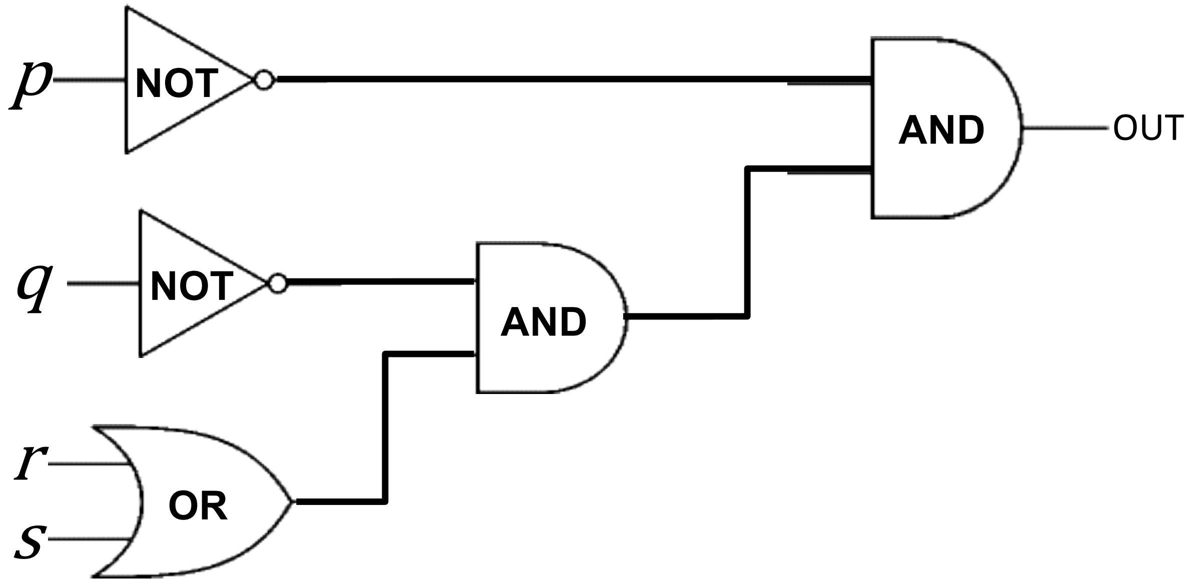
NOT Gate



p	OUT
1	0
0	1

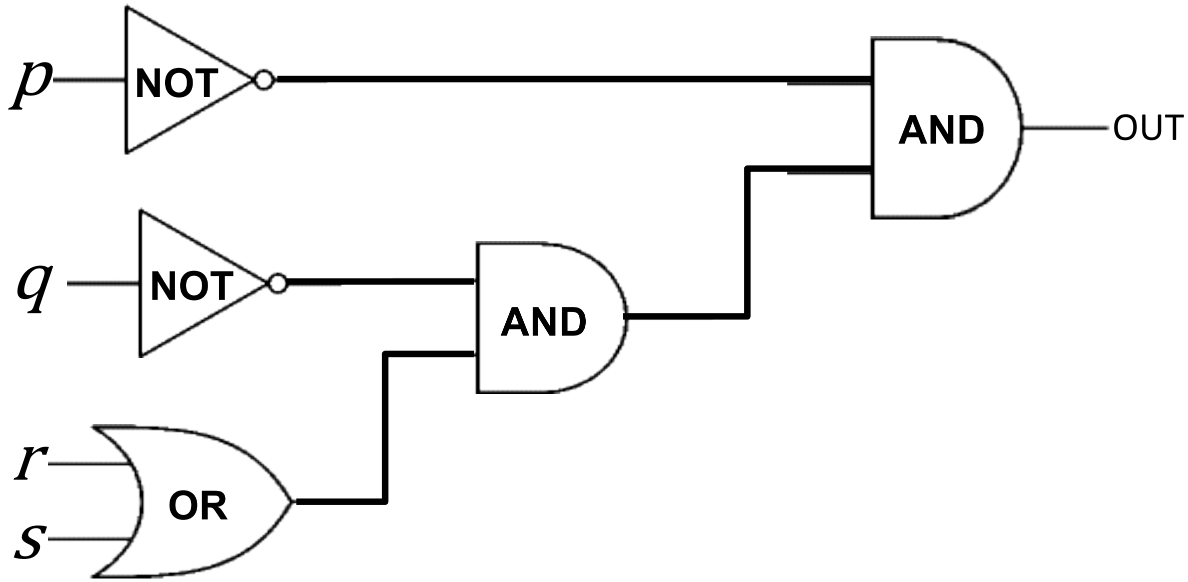
p	$\neg p$
T	F
F	T

Combinational Logic Circuits



Values get sent along wires connecting gates

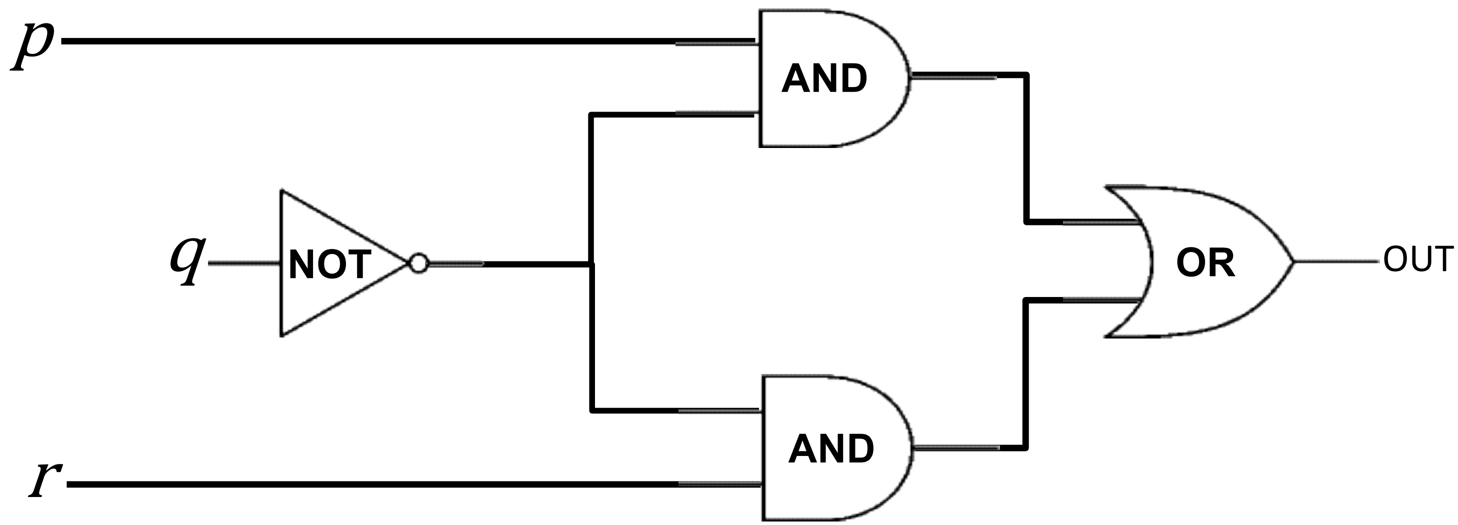
Combinational Logic Circuits



Values get sent along wires connecting gates

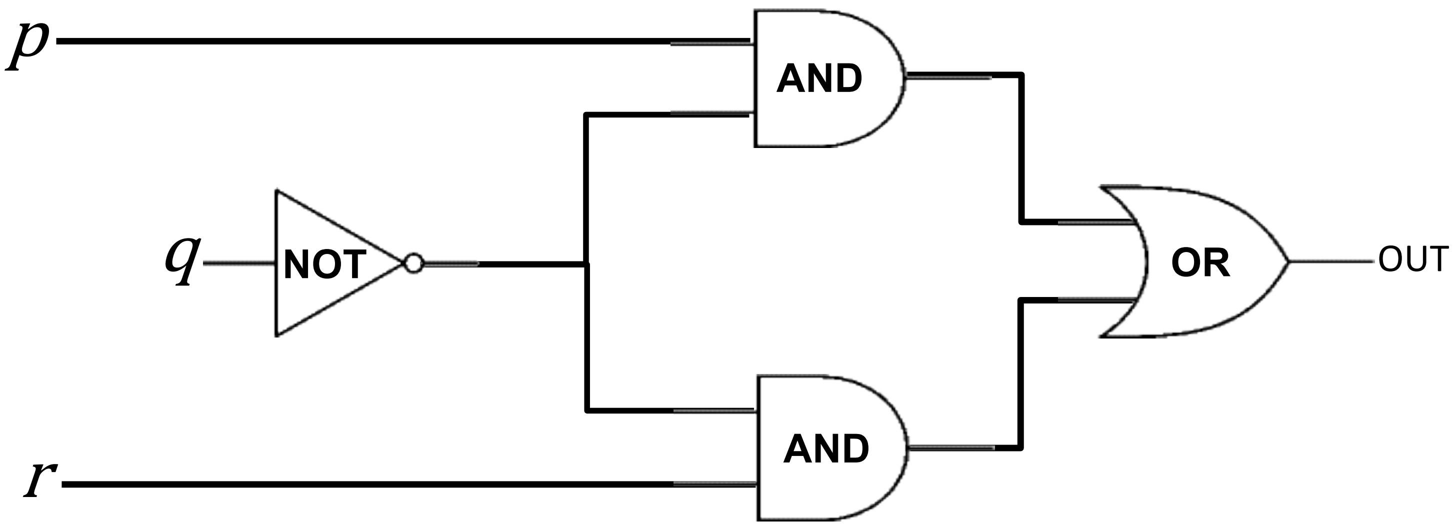
$$\neg p \land (\neg q \land (r \lor s))$$

Combinational Logic Circuits



Wires can send one value to multiple gates!

Combinational Logic Circuits



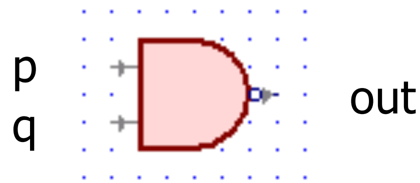
Wires can send one value to multiple gates!

$$(p \land \neg q) \lor (\neg q \land r)$$

Other Useful Gates

NAND

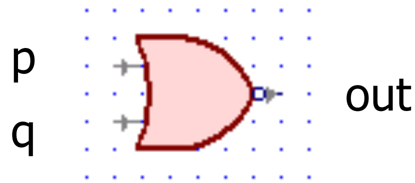
$\neg(p \wedge q)$



p	q	out
0	0	1
0	1	1
1	0	1
1	1	0

NOR

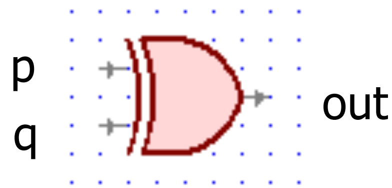
$\neg(p \vee q)$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	0

XOR

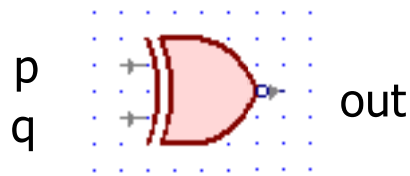
$p \bigoplus q$



p	q	out
0	0	0
0	1	1
1	0	1
1	1	0

XNOR

$p \leftrightarrow q$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	1

Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \vee \neg p$

$p \oplus p$

$(p \rightarrow q) \wedge p$

Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \lor \neg p$

This is a tautology. It's called the "law of the excluded middle".
If p is true, then $p \lor \neg p$ is true. If p is false, then $p \lor \neg p$ is true.

$p \oplus p$

This is a contradiction. It's always false no matter what truth value p takes on.

$(p \rightarrow q) \land p$

This is a contingency. When p=T, q=T, $(T \rightarrow T) \land T$ is true.
When p=T, q=F, $(T \rightarrow F) \land T$ is false.

Logical Equivalence

A = B means A and B are identical "strings":

– $p \wedge q = p \wedge q$

– $p \wedge q \neq q \wedge p$

Logical Equivalence

A = B means A and B are identical "strings":

- $p \wedge q = p \wedge q$

  These are equal, because they are character-for-character identical.

- $p \wedge q \neq q \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

A ≡ B means A and B have identical truth values:

- $p \wedge q \equiv p \wedge q$
- $p \wedge q \equiv q \wedge p$
- $p \wedge q \not\equiv q \vee p$

Logical Equivalence

A = B means A and B are identical "strings":

- $p \wedge q = p \wedge q$

  These are equal, because they are character-for-character identical.

- $p \wedge q \neq q \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

A ≡ B means A and B have identical truth values:

- $p \wedge q \equiv p \wedge q$

  Two formulas that are **equal** also are equivalent.

- $p \wedge q \equiv q \wedge p$

  These two formulas have the same truth table!

- $p \wedge q \not\equiv q \vee p$

  When p=T and q=F, $p \wedge q$ is false, but $p \vee q$ is true!

$A \leftrightarrow B$ vs. $A \equiv B$

$A \equiv B$ is an **assertion over all possible truth values** that A and B always have the same truth values.

$A \leftrightarrow B$ is a **proposition** that may be true or false depending on the truth values of the variables in A and B.

$A \equiv B$ and $(A \leftrightarrow B) \equiv \mathbf{T}$ have the same meaning.