CSE 311: Foundations of Computing I

Homework 4 (due Friday, October 25th at 11:00 PM)

Directions: *Write up carefully argued solutions to the following problems. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. However, you may use results from lecture, the theorems handout, and previous homeworks without proof.*

Note: Problems asking you to "prove" a claim require an English (not formal) proof. Within an English proof, however, it is okay to write out a chain of equivalences in table form, with explanations for each substitution along the side, as in previous assignments, rather than writing those steps with sentences.

1. Cat On a Hot Tin Proof (8 points)

Let the domain of discourse be foods, pets (including Garfield and Odie), and days of the week. We define the predicates $\mathsf{Cat}(x)$, $\mathsf{Dog}(x)$, and $\mathsf{Lasagna}(x)$ to mean that x is a cat, a dog, or a lasagna, respectively. We also define the predicates $\mathsf{Loves}(x, y)$ and $\mathsf{Hates}(x, y)$ to mean that x loves y and x hates y, respectively.

Prove the following claim:

$$\forall x\,(\mathsf{Lasagna}(x) \to \exists y\,(\mathsf{Cat}(y) \land \mathsf{Loves}(y, x) \land \mathsf{Hates}(y, \mathsf{Monday}) \land \neg\forall z\,(\mathsf{Dog}(z) \to \mathsf{Hates}(z, y))))$$

Common-knowledge facts about about Garfield and Odie do not require any proof and may simply be stated. If you are unfamiliar with that comic, the Wikipedia article may be helpful.

2. Teacher's Set (24 points)

Let A, B, and C be sets. Prove or disprove the following claims.

For the proofs, you may not cite the Meta Theorem. However, you should feel free to follow it as a template for how to write your own proof.

(a) $A \cap \overline{(B \cap \overline{A})} = A$

(b) $(B \setminus A) \cap (C \setminus A) = (B \cup C) \setminus A$

(c) $(B \cup \overline{A}) \cap (B \cup \overline{C}) = B \cup \overline{(C \cup A)}$

3. April Showers Bring May Powers (24 points)

Let S and T be sets. Prove or disprove the following claims.

(a) $\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T) \cup \mathcal{P}(S \cap T)$

(b) $\mathcal{P}(S \cap T) = \{\emptyset\} \cup (\mathcal{P}(S) \setminus \mathcal{P}(S \setminus T))$

(c) $\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T)$

4. Keeping Up With the Cartesians (24 points)

Let A, B, and C be sets.

(a) [12 Points] Prove that, if $B \subseteq \emptyset$, then $A \times B = B \times C$.

(If it helps, recall that $\emptyset$ is defined by the fact that $\forall x\, (x \notin \emptyset)$ holds.)

(b) [12 Points] Prove that, if $B \not\subseteq \emptyset$ and $A \times B \subseteq B \times C$, then $A \subseteq B$.

5. A Good Prime Was Had By All (20 points)

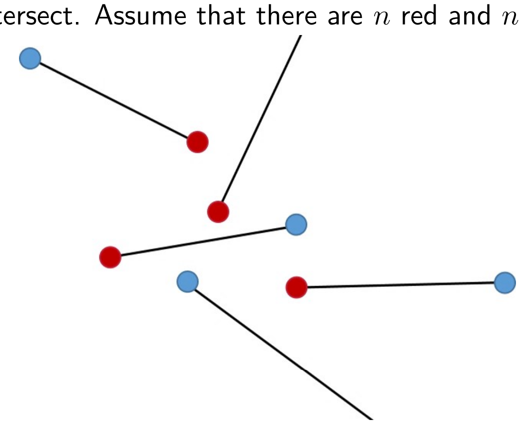
Prove that, if $p > 2$ is prime and $p \not\equiv 0 \pmod 3$, then $p \bmod 12 \in \{1, 5, 7, 11\}$.

Hint 1: Note that $p \bmod 12$ has only 12 possible values $(0 \ldots 11)$.

Hint 2: Consider the proof strategies discussed in lecture.

6. Extra Credit: Match-22 (0 points)

In this problem, you will show that given n red points and n blue points in the plane such that no three points lie on a common line, it is possible to draw line segments between red-blue pairs so that all the pairs are matched and none of the line segments intersect. Assume that there are n red and n blue points fixed in the plane.



A *matching* M is a collection of n line segments connecting distinct red-blue pairs. The *total length* of a matching M is the sum of the lengths of the line segments in M. Say that a matching M is *minimal* if there is no matching with a smaller total length.

Let IsMinimal(M) be the predicate that is true precisely when M is a minimal matching. Let HasCrossing(M) be the predicate that is true precisely when there are two line segments in M that cross each other.

Give an argument in English explaining why there must be at least one matching M so that IsMinimal(M) is true, i.e.

$$\exists M \text{IsMinimal}(M))$$

Give an argument in English explaining why

$$\forall M(\text{HasCrossing}(M) \rightarrow \neg\text{IsMinimal}(M))$$

Then, use the two results above to give a proof of the statement:

$$\exists M \neg\text{HasCrossing}(M).$$