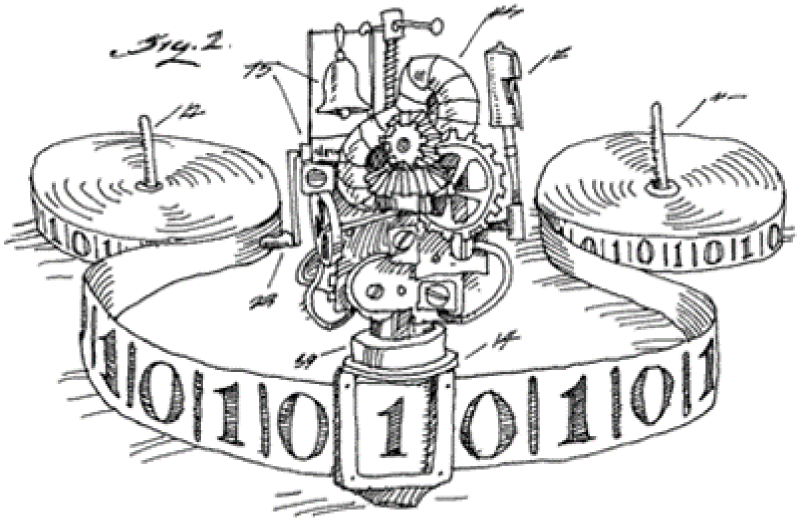
# CSE 311: Foundations of Computing

## Lecture 28: Undecidability, Reductions, and Turing Machines

HW7 solutions handed out in section yesterday

HW8 solutions coming today

Don't leave without them.



# Final exam

- **Monday** at either **2:30-4:20 p.m.** or **4:30-6:20 p.m.**
  - **Sieg Hall 134**
  - You need to fill out **Catalyst Survey** to say which you are taking by **midnight Sunday** night.
  - Bring your **UW ID** and have it out and ready during the exam
- **Comprehensive** coverage. If you had a homework question on it, it is fair game. See link on webpage.
  - Includes pre-midterm topics, e.g. formal proofs. Will contain the same sheets at end.
- **Review session: Sunday 3:30-5:00 p.m. EEB 105**
  - **Bring your questions !!**

# Review: Countability vs Uncountability

- To prove a set A countable you must show
  - There exists a listing $x_1,x_2,x_3$, … such that every element of A is in the list.

- To prove a set B uncountable you must show
  - For *every* listing $x_1,x_2,x_3$, … there exists some element in B that is not in the list.

  - The diagonalization proof shows how to describe a missing element d in B based on the listing $x_1,x_2,x_3$, … .
    *Important:* the proof produces a d no matter what the listing is.

# Last time: Undecidability of the Halting Problem

CODE(**P**) means "the code of the program **P**"

**The Halting Problem**

**Given:** - CODE(**P**) for any program **P**
- input **x**

**Output:** **true** if **P** halts on input **x**
**false** if **P** does not halt on input **x**

**Theorem** [Turing]: **There is no program that solves the Halting Problem**

**Proof:** By contradiction.

Assume that a program **H** solving the Halting program does exist. Then program **D** must exist

Does **D**(CODE(**D**)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don’t halt */
    }
    else {
        return;          /*    halt    */
    }
}
```

**H** solves the halting problem implies that

H(CODE(**D**),x) is **true** iff **D**(x) halts,  H(CODE(D ot

Suppose that **D**(CODE(**D**)) **halts**.

Then, by definition of **H** it mus

H(CODE(**D**), CO

Which by the defi s **D**(CODE(**D**)) **doesn’t halt**

Suppose th **doesn’t halt**.

Th of **H** it must be that

ODE(**D**), CODE(**D**)) is **false**

Whi by the definition of **D** means **D**(CODE(**D**)) **halts**

**The ONLY assumption was the program H exists so that assumption must have been false.**

**Contradiction!**

# SCOOPING THE LOOP SNOOPER
## A proof that the Halting Problem is undecidable

by Geoffrey K. Pullum (U. Edinburgh)

*No general procedure for bug checks succeeds.*
Now, I won’t just assert that, I’ll show where it leads:
I will prove that although you might work till you drop,
you cannot tell if computation will stop.

For imagine we have a procedure called *P*
that for specified input permits you to see
whether specified source code, with all of its faults,
defines a routine that eventually halts.

You feed in your program, with suitable data,
and *P* gets to work, and a little while later
(in finite compute time) correctly infers
whether infinite looping behavior occurs...

# SCOOPING THE LOOP SNOOPER

…

Here's the trick that I'll use – and it's simple to do.
I'll define a procedure, which I will call *Q*,
that will use *P*'s predictions of halting success
to stir up a terrible logical mess.

…

And this program called *Q* wouldn't stay on the shelf;
I would ask it to forecast its run on *itself*.
When it reads its own source code, just what will it do?
What's the looping behavior of *Q* run on *Q*?

…

Full poem at:

http://www.lel.ed.ac.uk/~gpullum/loopsnoop.html

# The Halting Problem isn't the only hard problem

- Can use the fact that the Halting Problem is undecidable to show that other problems are undecidable

General method:

Prove that if there were a program deciding B then there would be a way to build a program deciding the Halting Problem.

"B decidable → Halting Problem decidable"

Contrapositive:

"Halting Problem undecidable → B undecidable"

Therefore B is undecidable

# Last time: A CSE 141 assignment

Students should write a Java program that:

- Prints “Hello” to the console
- Eventually exits

GradeIt, PracticeIt, etc. need to grade the students.

How do we write that grading program?

WE CAN’T: THIS IS IMPOSSIBLE!

# A related undecidable problem

- **HelloWorldTesting Problem:**
  - **Input: CODE(Q)** and **x**
  - **Output:**

    **True** if **Q** outputs "HELLO WORLD" on input **x**

    **False** if **Q** does not output "HELLO WORLD" on input **x**

- **Theorem: The HelloWorldTesting Problem is undecidable.**
- **Proof idea: Show that if there is a program T to decide HelloWorldTesting then there is a program H to decide the Halting Problem for code(P) and x.**

# A related undecidable problem

- **Suppose there is a program T that solves the HelloWorldTesting problem. Define program H that takes input CODE(P) and x and does the following:**
  - **Creates CODE(Q) from CODE(P) by**

    **(1) removing all output statements from CODE(P), and**

    **(2) adding a** System.out.println("HELLO WORLD") **immediately before any spot where P could halt**

    **Then runs T on input CODE(Q) and x.**

- If **P halts on input x** then **Q** prints HELLO WORLD and halts and so H outputs **true** (because T outputs true on input **CODE(Q)**)
- If **P doesn't halt on input x** then **Q** won't print anything since we removed any other print statement from CODE(**Q**) so H outputs **false**

**We know that such an H cannot exist. Therefore T cannot exist.**

# The HaltsNoInput Problem

- **Input: CODE(R)** for program **R**
- **Output: True** if **R** halts without reading input

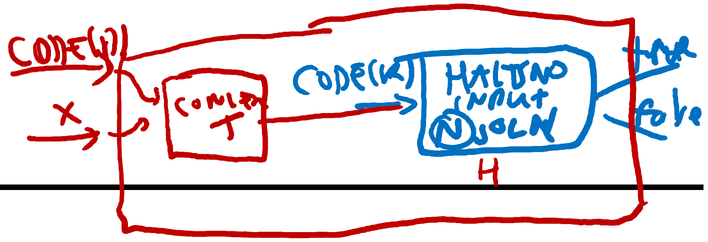
  **False** otherwise.

**Theorem: HaltsNoInput is undecidable**

**General idea "hard-coding the input":**

- Show how to use **CODE(P)** and **x** to build **CODE(R)** so **P halts on input x ⇔ R halts without reading input**

# The HaltsNoInput Problem



## “Hard-coding the input”:

- Show how to use **CODE(P)** and **x** to build **CODE(R)** so **P halts on input x ⇔ R halts without reading input**

- Replace input statement in **CODE(P)** that reads input **x** into variable **var**, by a hard-coded assignment statement:

  **var = x**

  to produce **CODE(R)**.

- So if we have a program **N** to decide **HaltsNoInput** then we can use it as a subroutine as follows to decide the Halting Problem, which we know is impossible:
  - On input CODE(**P**) and **x**, produce CODE(**R**). Then run **N** on input CODE(~~Q~~R) and output the answer that **N** gives.

- The impossibility of writing the CSE 141 grading program follows by combining the ideas from the undecidability of HaltsNoInput and HelloWorld.

# More Reductions

- Can use undecidability of these problems to show that other problems are undecidable.

- For instance:

$\mathrm{EQUIV}(P, Q)$ : **True** if $P(x)$ and $Q(x)$ have the same behavior for every input $x$

↑

**False** otherwise

Q : run forever ) ( false

always halts

outputs true

Q : print "HELLO WORLD" ✓

# Rice's theorem

Not *every* problem on programs is undecidable!

Which of these is decidable?

- Input `CODE(P)` and x
  Output: true  if P prints "ERROR" on input x
                after less than 100 steps
            false otherwise

- Input `CODE(P)` and x
  Output: true  if P prints "ERROR" on input x
                after more than 100 steps
            false otherwise

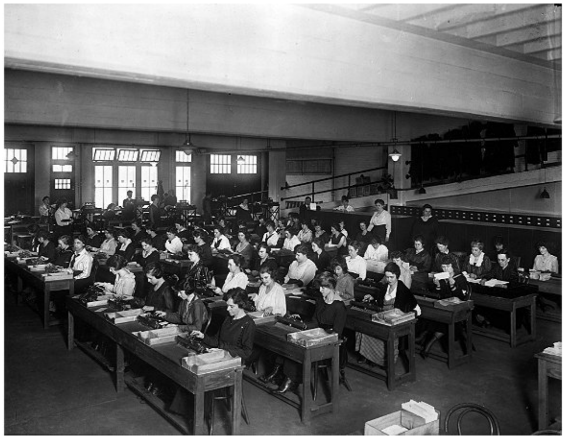
Rice's Theorem (a.k.a. Compilers Suck Theorem - informal):

Any "non-trivial" property of the input-output behavior of Java programs is undecidable.

# Computers and algorithms

- Does Java (or any programming language) cover all possible computation? Every possible algorithm?

- There was a time when computers were people who did calculations on sheets paper to solve computational problems



- Computers as we known them arose from trying to understand everything these people could do.

# Before Java

## 1930's:

How can we formalize what algorithms are possible?

- **Turing machines** (Turing, Post)
  - basis of modern computers
- **Lambda Calculus** (Church)
  - basis for functional programming, LISP
- **$\mu$-recursive functions** (Kleene)
  - alternative functional programming basis

# Turing machines

**Church-Turing Thesis:**

Any reasonable model of computation that includes all possible algorithms is equivalent in power to a Turing machine

Evidence

- Intuitive justification
- Huge numbers of models based on radically different ideas turned out to be equivalent to TMs

# Turing machines

- **Finite Control**
  - **Brain/CPU** that has only a **finite # of possible "states of mind"**
- **Recording medium**
  - **An unlimited supply of** blank "**scratch paper**" on which to **write & read symbols,** each chosen from a **finite set of possibilities**
  - **Input** also supplied **on** the **scratch paper**
- **Focus of attention**
  - **Finite control** can only **focus on** a **small portion of the recording medium at once**
  - **Focus of attention** can only **shift a small amount at a time**

# Turing machines

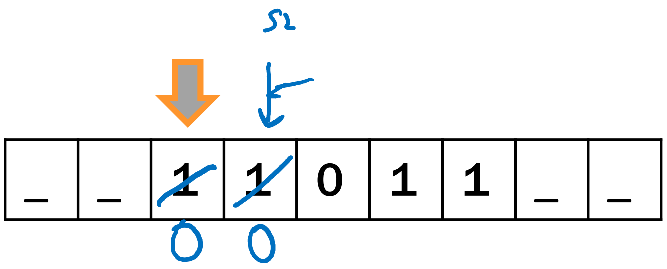
- Recording medium
  - An **infinite** read/write "**tape**" marked off into **cells**
  - Each **cell** can store **one symbol or** be "**blank**"
  - **Tape** is **initially** all **blank except** a few **cells** of the tape **containing the input string**
  - **Read/write head** can **scan one cell** of the tape - **starts on input**

- In each step, a Turing machine
  1. **Reads** the **currently scanned cell**
  2. **Based on current state and scanned symbol**
     1. **Overwrites symbol in scanned cell**
     2. **Moves read/write head left or right one cell**
     3. **Changes to a new state**

- **Each Turing Machine is specified by its finite set of rules**

# Turing machines

| | _ | 0 | 1 |
|---|---|---|---|
| $s_1$ | (1, L, $s_3$) | (1, L, $s_4$) | (0, R, $s_2$) |
| $s_2$ | (0, R, $s_1$) | (1, R, $s_1$) | (0, R, $s_1$) |
| $s_3$ | | | |
| $s_4$ | | | |



# UW CSE's Steam-Powered Turing Machine

ACCEPT
REJECT

Original in Sieg Hall stairwell

# Turing machines

**Ideal Java/C programs:**

- Just like the Java/C you're used to programming with, except you never run out of memory
  - Constructor methods always succeed
  - **malloc** in C never fails

**Equivalent to Turing machines except a lot easier to program:**

- Turing machine definition is useful for breaking computation down into simplest steps
- We only care about high level so we use programs

# Turing's big idea part 1: Machines as data

**Original Turing machine definition:**

- A different "machine" **M** for each task
- Each machine **M** is defined by a finite set of possible operations on finite set of symbols
- So... **M** has a finite description as a sequence of symbols, its "code", which we denote **<M>**

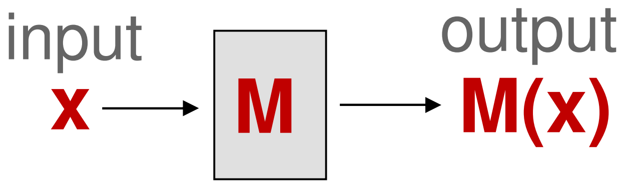
**You already are used to this idea with the notion of the program code or text but this was a new idea in Turing's time.**

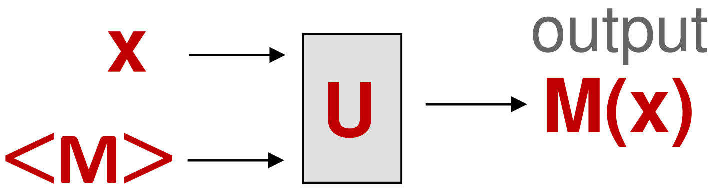
# Turing's big idea part 2:  A Universal TM

- **A Turing machine interpreter U**
  - On input <M> and its input x,

    U outputs the same thing as M does on input x
  - At each step it decodes which operation M would have performed and simulates it.
- **One Turing machine is enough**
  - Basis for modern stored-program computer

    Von Neumann studied Turing's UTM design





# Takeaway from undecidability

- You can’t rely on the idea of improved compilers and programming languages to eliminate major programming errors
  - truly safe languages can’t possibly do general computation
- Document your code
  - there is no way you can expect someone else to figure out what your program does with just your code; since in general it is provably impossible to do this!

# We’ve come a long way!

- Propositional Logic.
- Boolean logic and circuits.
- Boolean algebra.
- Predicates, quantifiers and predicate logic.
- Inference rules and formal proofs for propositional and predicate logic.
- English proofs.
- Set theory.
- Modular arithmetic.
- Prime numbers.
- GCD, Euclid's algorithm, modular inverse, and exponentiation.

# We’ve come a long way!

- Induction and Strong Induction.
- Recursively defined functions and sets.
- Structural induction.
- Regular expressions.
- Context-free grammars and languages.
- Relations and composition.
- Transitive-reflexive closure.
- Graph representation of relations and their closures.

# We’ve come a long way!

- DFAs, NFAs and language recognition.
- Product construction for DFAs.
- Finite state machines with outputs at states.
- Minimization algorithm for finite state machines
- Conversion of regular expressions to NFAs.
- Subset construction to convert NFAs to DFAs.
- Equivalence of DFAs, NFAs, Regular Expressions
- Finite automata for pattern matching.
- Method to prove languages not accepted by DFAs.
- Cardinality, countability and diagonalization
- Undecidability: Halting problem and evaluating properties of programs.

# What’s next? …after the final exam…

- **Foundations II** (312)
  - Fundamentals of counting, discrete probability, applications of randomness to computing, statistical algorithms and analysis
  - Ideas critical for machine learning, algorithms

- **Data Abstractions** (332)
  - Data structures, a few key algorithms, parallelism
  - Brings programming and theory together
  - Makes heavy use of induction and recursive defns

# Course Evaluation Online

- Fill this out by Sunday night!
  - Your ability to fill it out will disappear at 11:59 p.m. on Sunday.
  - It will be worth your while to fill it out!

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