CSE 311: Foundations of Computing

Lecture 27: Undecidability

Last time: Countable sets

A set S is **countable** iff we can order the elements of S as

$$S = \{x_1, x_2, x_3, \ldots\}$$

$\frac{1}{1}\quad \frac{2}{1}\quad \frac{3}{1}\quad \frac{4}{1}\quad \frac{5}{1}$

Countable sets:

$\mathbb{N}$ - the natural numbers

$\mathbb{Z}$ - the integers

$\frac{p}{q}$ $\quad\mathbb{Q}$ - the rationals

Σ^*- the strings over any finite Σ

The set of all Java programs

Shown by "dovetailing"

$$S = G_1 \cup G_2 \cup G_3 \cup \cdots$$

with each G_i is finite

Last time: Not every set is countable

Theorem [Cantor]:
The set of real numbers between 0 and 1 is not countable.

Proof using "diagonalization".

Last time: Proof that $[0,1)$ is not countable

Suppose, for the sake of contradiction, that there is a list of them:

		1	2	3	4						
r_1	0.	5^{1}	0	0	0						
r_2	0.	3	3^{5}	3	3						
r_3	0.	1	4	2^{5}	8	5	7	1	4	...	...
r_4	0.	1	4	1	5^{1}	9	2	6	5	...	...
						2^{5}	1	2	2	...	...
						0	0^{5}	0	0	...	...
						8	1	8^{5}	2	...	...

Flipping rule:

If digit is 5, make it 1.

If digit is not 5, make it 5.

For every $n \geq 1$:

$r_n \neq d = 0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55}\cdots$

because the numbers differ on the n-th digit!

So the list is incomplete, which is a contradiction.

Thus the real numbers between 0 and 1 are **not countable**: “uncountable”

■

A note on this proof

- The set of rational numbers in [0,1) also have decimal representations like this
  - The only difference is that rational numbers always have repeating decimals in their expansions 0.33333… or .25000000…
- So why wouldn't the same proof show that this set of rational numbers is uncountable?
  - Given any listing (even one that is good like the dovetailing listing) we could create the flipped diagonal number d as before
  - However, d would not have a repeating decimal expansion and so wouldn't be a rational #

    It would not be a "missing" number, so no contradiction.

Last time:

The set of all functions $f : \mathbb{N} \to \{0, \dots, 9\}$ is uncountable

Supposed listing of all the functions:

	1	2	3	4						
f_1	5^{1}	0	0	0						
f_2	3	3^{5}	3	3						
f_3	1	4	2^{5}	8	5	7	1	4	...	...
f_4	1	4	1	5^{1}	9	2	6	5	...	...
f_5	1	2	1	2	2^{5}	1	2	2	...	...
f_6	2	5	0	0	0	0^{5}	0	0	...	...
f_7	7	1	8	2	8	1	8^{5}	2	...	...

Flipping rule:

If $f_n(n) = 5$, set $D(n) = 1$

If $f_n(n) \neq 5$, set $D(n) = 5$

For all n, we have $D(n) \neq f_n(n)$. Therefore $D \neq f_n$ for any n and the list is incomplete! $\Rightarrow$ $\{f \mid f: \mathbb{N} \to \{0,1, \dots, 9\}\}$ is **not** countable

Last time: Uncomputable functions

We have seen that:

- The set of all (Java) programs is countable
- The set of all functions $f : \mathbb{N} \to \{0, \ldots, 9\}$ is not countable

So: There must be some function $f : \mathbb{N} \to \{0, \ldots, 9\}$ that is not computable by any program!

Interesting… maybe.

Can we come up with an explicit function that is uncomputable?

A “Simple” Program

```
public static void collatz(n) {
    if (n == 1) {
        return 1;
    }
    if (n % 2 == 0) {
        return collatz(n/2)
    }
    else {
        return collatz(3*n + 1)
    }
}
```

11
34
17
52
26
13
40
20
10
5
16
8
4
2
1

What does this program do?

… on n=11?

… on n=1000000000000000001?

A “Simple” Program

```
public static void collatz(n) {
    if (n == 1) {
        return 1;
    }
    if (n % 2 == 0) {
        return collatz(n/2)
    }
    else {
        return collatz(3*n + 1)
    }
}
```

Nobody knows whether or not this program halts on all inputs!

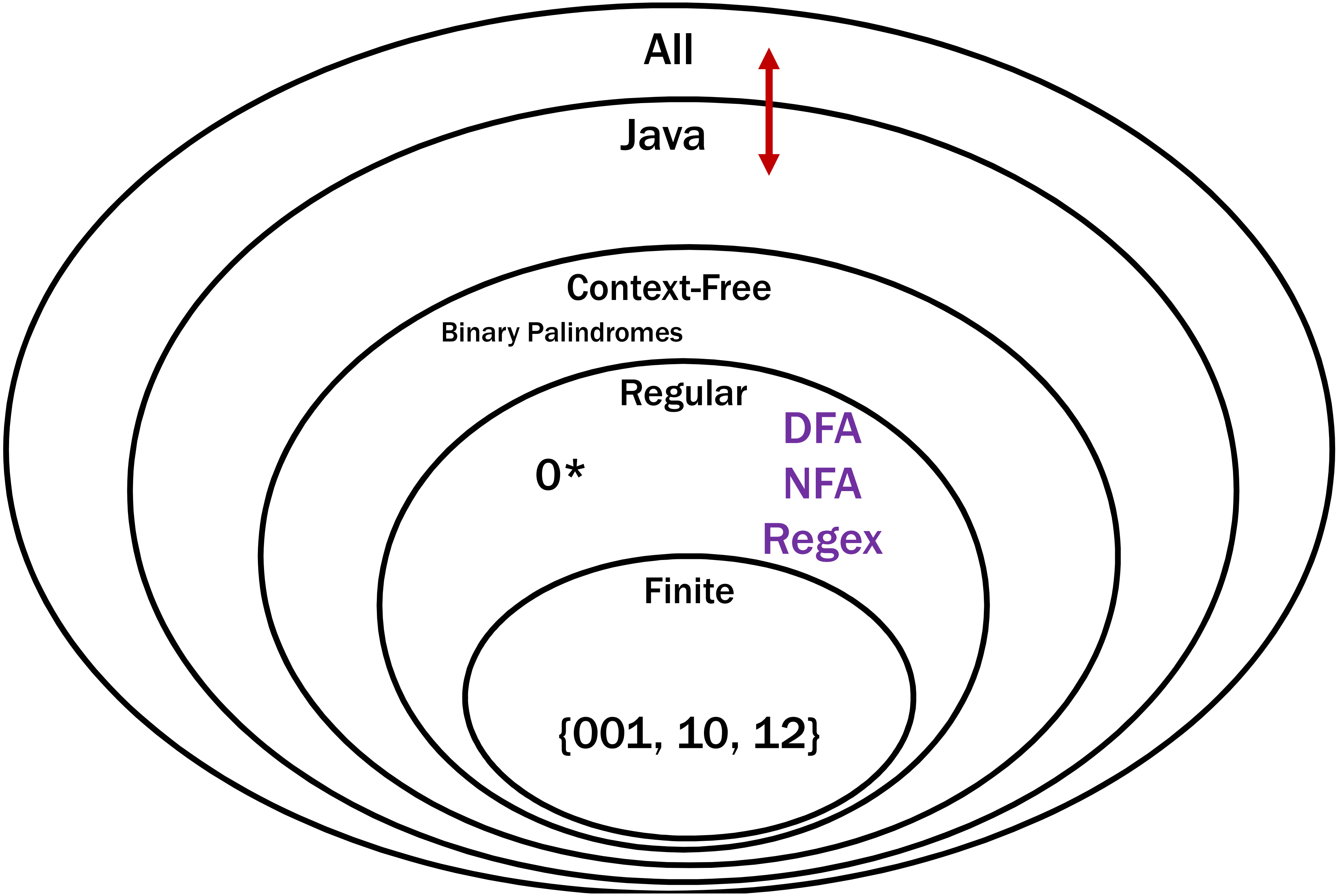
Trying to solve this has been called a “mathematical disease”.

What does this program do?

… on `n=11`?

… on `n=100000000000000000001`?

Recall our language picture



Some Notation

We're going to be talking about *Java code*.

CODE(P) will mean "the code of the program **P**"

So, consider the following function:

P("fun")

```
public String P(String x) {
    return new String(Arrays.sort(x.toCharArray());
}
```

What is **P(CODE(P))**?

"(((())))..;AACPSSaaabceegghiiiilnnnnnooprrrrrrrrrrrsssttttttuuwxxyy{}"

The Halting Problem

CODE(**P**) means "the code of the program **P**"

The Halting Problem

Given: - CODE(**P**) for any program **P**
- input **x**

Output: **true** if **P** halts on input **x**
false if **P** does not halt on input **x**

Undecidability of the Halting Problem

CODE(**P**) means "the code of the program **P**"

> **The Halting Problem**
>
> **Given:** - CODE(**P**) for any program **P**
> - input **x**
>
> **Output:** **true** if **P** halts on input **x**
> **false** if **P** does not halt on input **x**

Theorem [Turing]: **There is no program that solves the Halting Problem**

Proof by contradiction

- Suppose that H is a Java program that solves the Halting problem.   Then we can write this program:

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true);    /* don't halt */
    }
    else {
        return;          /*    halt    */
    }
}
```

- Does D(CODE(D)) halt?

Does **D**(CODE(**D**)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;        /*   halt   */
    }
}
```

Does **D**(CODE(**D**)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;        /*    halt    */
    }
}
```

H solves the halting problem implies that
H(CODE(**D**),x) is **true** iff **D**(x) halts, **H**(CODE(**D**),x) is **false** iff not

Note: Even though the program **D** has a **while**(true), that doesn't mean that the program **D** actually goes into an infinite loop on input **x**, which is what **H** has to determine

Does **D**(CODE(**D**)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;        /*    halt    */
    }
}
```

H solves the halting problem implies that

H(CODE(**D**),x) is **true** iff **D**(x) halts, **H**(CODE(**D**),x) is **false** iff not

Suppose that **D**(CODE(**D**)) **halts.**

$x = \text{CODE}(D)$

Then, by definition of **H** it must be that

H(CODE(**D**), CODE(**D**)) is **true**

Which by the definition of **D** means **D**(CODE(**D**)) **doesn't halt**

$P \wedge \neg P \equiv F$

P is "D(CODE(D)) halts"

Does **D**(CODE(**D**)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;       /*   halt   */
    }
}
```

H solves the halting problem implies that
H(CODE(**D**),x) is **true** iff **D**(x) halts, **H**(CODE(**D**),x) is **false** iff not

Suppose that **D**(CODE(**D**)) **halts**.
Then, by definition of **H** it must be that
H(CODE(**D**), CODE(**D**)) is **true**
Which by the definition of **D** means **D**(CODE(**D**)) **doesn't halt**

F

x = CODE(D)

Suppose that **D**(CODE(**D**)) **doesn't halt**.
Then, by definition of **H** it must be that
H(CODE(**D**), CODE(**D**)) is **false**
Which by the definition of **D** means **D**(CODE(**D**)) **halts**

$\neg P \wedge P \equiv F$

Does D(CODE(D)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;        /*    halt    */
    }
}
```

H solves the halting problem implies that

H(CODE(D),x) is **true** iff D(x) halts, H(CODE(D) … ot

Suppose that D(CODE(D)) **halts**.

Then, by definition of H it mus…

H(CODE(D), CO…

Which by the defi… D(CODE(D)) **doesn't halt**

Suppose th… **doesn't halt**.

Th… of H it must be that

…ODE(D), CODE(D)) is **false**

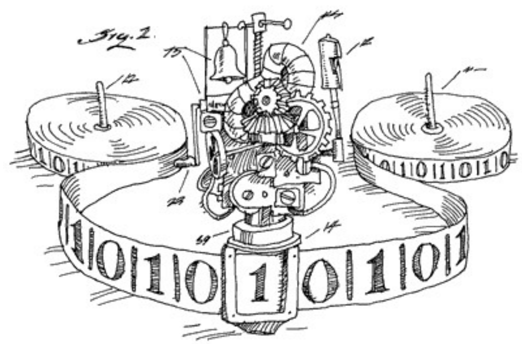
Whi… by the definition of D means D(CODE(D)) **halts**

Contradiction!

The ONLY assumption was that the program H exists so that assumption must have been false.

Done

- We proved that there is no computer program that can solve the Halting Problem.
  - There was nothing special about Java*
    [Church-Turing thesis]



- This tells us that there is no compiler that can check our programs and guarantee to find any infinite loops they might have.

Where did the idea for creating D come from?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;          /*    halt    */
    }
}
```

D halts on input code(P) iff H(code(P),code(P)) outputs false
iff P doesn't halt on input code(P)

Therefore for any program P, D differs from P on input code(P)

Connection to diagonalization

Write <P> for CODE(P)

Some possible inputs **x**

	$\langle P_1\rangle$ $\langle P_2\rangle$ $\langle P_3\rangle$ $\langle P_4\rangle$ $\langle P_5\rangle$ $\langle P_6\rangle$ ....
P_1	
P_2	
P_3	
P_4	
P_5	
P_6	
P_7	
P_8	
P_9	
.	
.	

All programs **P**

This listing of all programs really does exist since the set of all Java programs is countable

The goal of this "diagonal" argument is not to show that the listing is incomplete but rather to show that a "flipped" diagonal element is not in the listing

Connection to diagonalization

Write <**P**> for CODE(**P**)

Some possible inputs **x**

All programs **P**

	$<P_1>$	$<P_2>$	$<P_3>$	$<P_4>$	$<P_5>$	$<P_6>$	....					
P_1	0	1	1	0	1	1	1	0	0	0	1	...
P_2	1	1	0	1	0	1	1	0	1	1	1	...
P_3	1	0	1	0	0	0	0	0	0	0	1	...
P_4	0	1	1	0	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1	0	1	0	...
P_9	. . . .											
.	. . . .											
.												

(P,x) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

Connection to diagonalization

Write <P> for CODE(P)

Some possible inputs **x**

All programs **P**

	$\langle P_1\rangle$	$\langle P_2\rangle$	$\langle P_3\rangle$	$\langle P_4\rangle$	$\langle P_5\rangle$	$\langle P_6\rangle$	....					
P_1	$0^{\mathbf{1}}$	1	1	0	1							
P_2	1	$1^{\mathbf{0}}$	0	1	0							
P_3	1	0	$1^{\mathbf{0}}$	0	0							
P_4	0	1	1	$0^{\mathbf{1}}$	1	0	1	1	0	1	0	...
P_5	0	1	1	1	$1^{\mathbf{0}}$	1	1	0	0	0	1	...
P_6	1	1	0	0	0	$1^{\mathbf{0}}$	1	0	1	1	1	...
P_7	1	0	1	1	0	0	$0^{\mathbf{1}}$	0	0	0	1	...
P_8	0	1	1	1	1	0	1	$1^{\mathbf{0}}$	0	1	0	...
P_9	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.
.												

Want behavior of program ***D*** to be like the flipped diagonal, so it can't be in the list of all programs.

(P,x) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

The Halting Problem isn't the only hard problem

- Can use the fact that the Halting Problem is undecidable to show that other problems are undecidable

General method:

Prove that if there were a program deciding B then there would be a way to build a program deciding the Halting Problem.

"B decidable → Halting Problem decidable" $P \to Q$

Contrapositive: $\neg Q \leftarrow \neg P$

"Halting Problem undecidable → B undecidable"

Therefore B is undecidable

A CSE 141 assignment

Students should write a Java program that:

- Prints "Hello" to the console
- Eventually exits

GradeIt, PracticeIt, etc. need to grade the students.

How do we write that grading program?

WE CAN'T: THIS IS IMPOSSIBLE!

A related undecidable problem

- **HelloWorldTesting Problem:**
  - **Input: CODE(Q)** and **x**
  - **Output:**

    True if **Q** outputs "HELLO WORLD" on input **x**

    False if **Q** does not output "HELLO WORLD" on input **x**

- **Theorem: The HelloWorldTesting Problem is undecidable.**
- **Proof idea: Show that if there is a program T to decide HelloWorldTesting then there is a program H to decide the Halting Problem for code(P) and x.**

A related undecidable problem

- **Suppose there is a program T that solves the HelloWorldTesting problem. Define program H that takes input CODE(P) and x and does the following:**
  - **Creates CODE(Q) from CODE(P) and x:**
    1) **Store reference to System.out then redirect to a StringWriter**
    2) **Call P(x)**
    3) **Print "Hello World"**
  - **Then runs T on input code(Q)**

- If **P halts on input x** then **Q** prints HELLO WORLD and halts and so **H** outputs **true** (because **T** outputs true on input **CODE(Q)**)
- If **P doesn't halt on input x** then **Q** won't print anything since we removed any other print statement from CODE(**Q**) so **H** outputs **false**

We know that such an H cannot exist. Therefore T cannot exist.