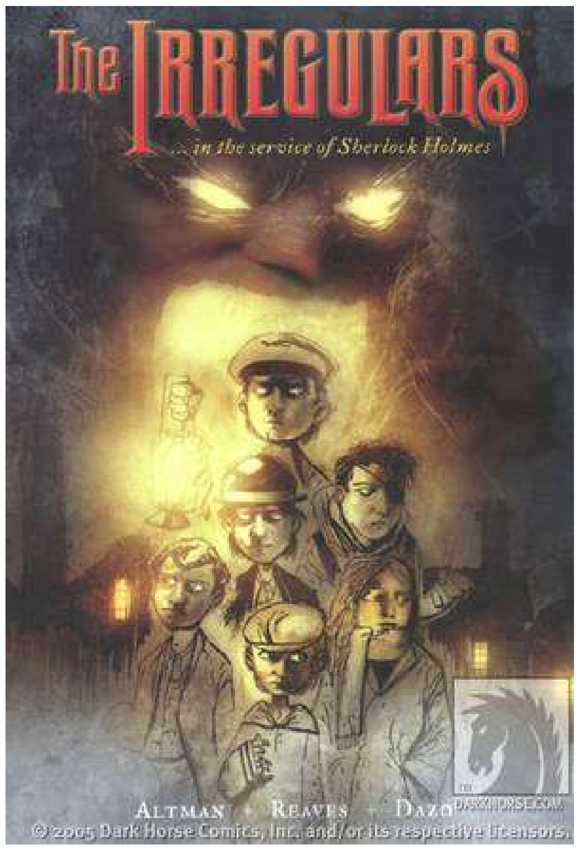
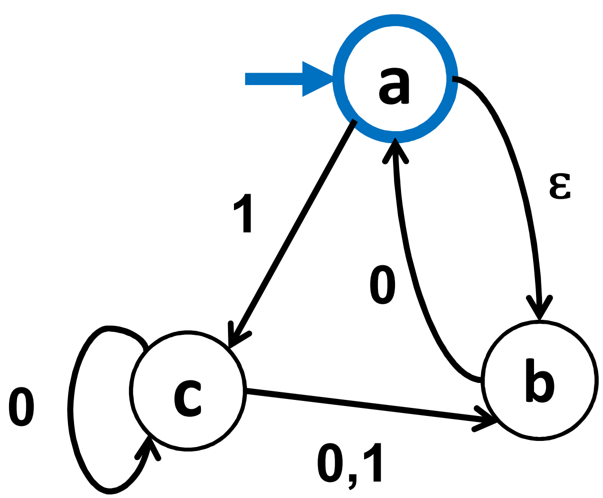
CSE 311: Foundations of Computing

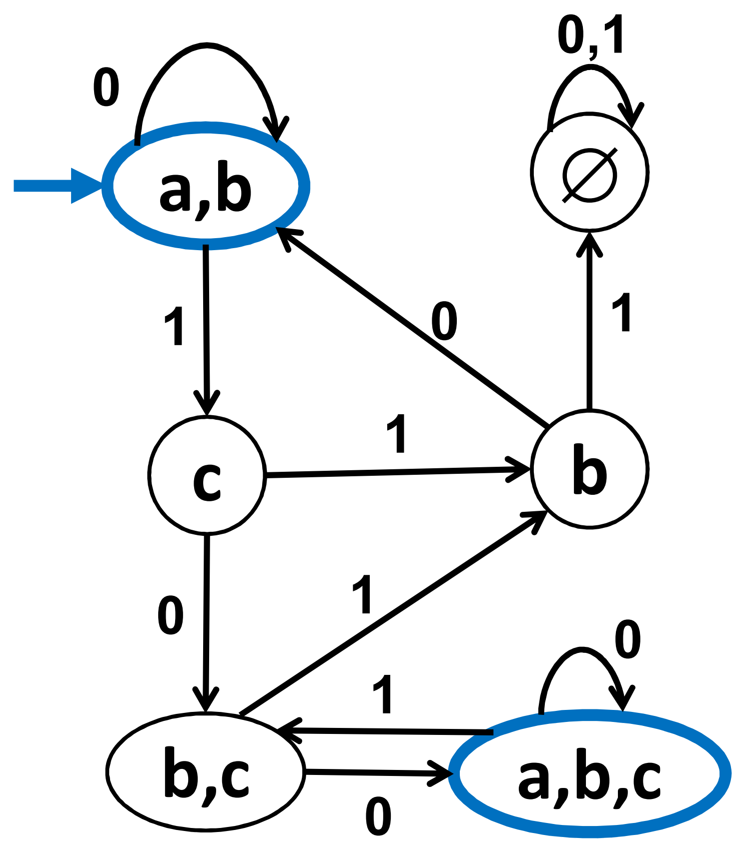
Lecture 25: Languages vs Representations: Limitations of Finite Automata and Regular Expressions



Last time: NFA to DFA



NFA



DFA

Exponential Blow-up in Simulating Nondeterminism

- In general the DFA might need a state for every subset of states of the NFA
  - Power set of the set of states of the NFA
  - n-state NFA yields DFA with at most 2^n states
  - We saw an example where roughly 2^n is necessary

    "Is the n^{th} char from the end a 1?"

The famous "P=NP?" question asks whether a similar blow-up is always necessary to get rid of non-determinism for polynomial-time algorithms

Last time: DFAs ≡ NFAs ≡ Regular expressions

We have shown how to build an optimal DFA for every regular expression

regular language

- Build NFA
- Convert NFA to DFA using subset construction
- Minimize resulting DFA

Theorem: A language is recognized by a DFA (or NFA) if and only if it has a regular expression

You need to know this fact but you don't need to know and we won't ask you anything about the construction for the "only if" direction from DFA/NFA to regular expression.

Application of FSMs: Pattern matching

- Given
  - a string s of n characters
  - a pattern p of m characters
  - usually $m \ll n$
- Find
  - all occurrences of the pattern p in the string s

- Obvious algorithm:
  - try to see if p matches at each of the positions in s stop at a failed match and try matching at the next position: $O(mn)$ running time in worst case

Application of FSMs: Pattern Matching

- With DFAs can do this in $O(m + n)$ time.

- Even more general idea in practice: implemented in regular expression pattern matchers like grep:
  - Convert regular expression pattern to an NFA
  - Start building the equivalent DFA from the NFA using the subset construction but do this "on the fly": only add arcs that are actually followed by the input text

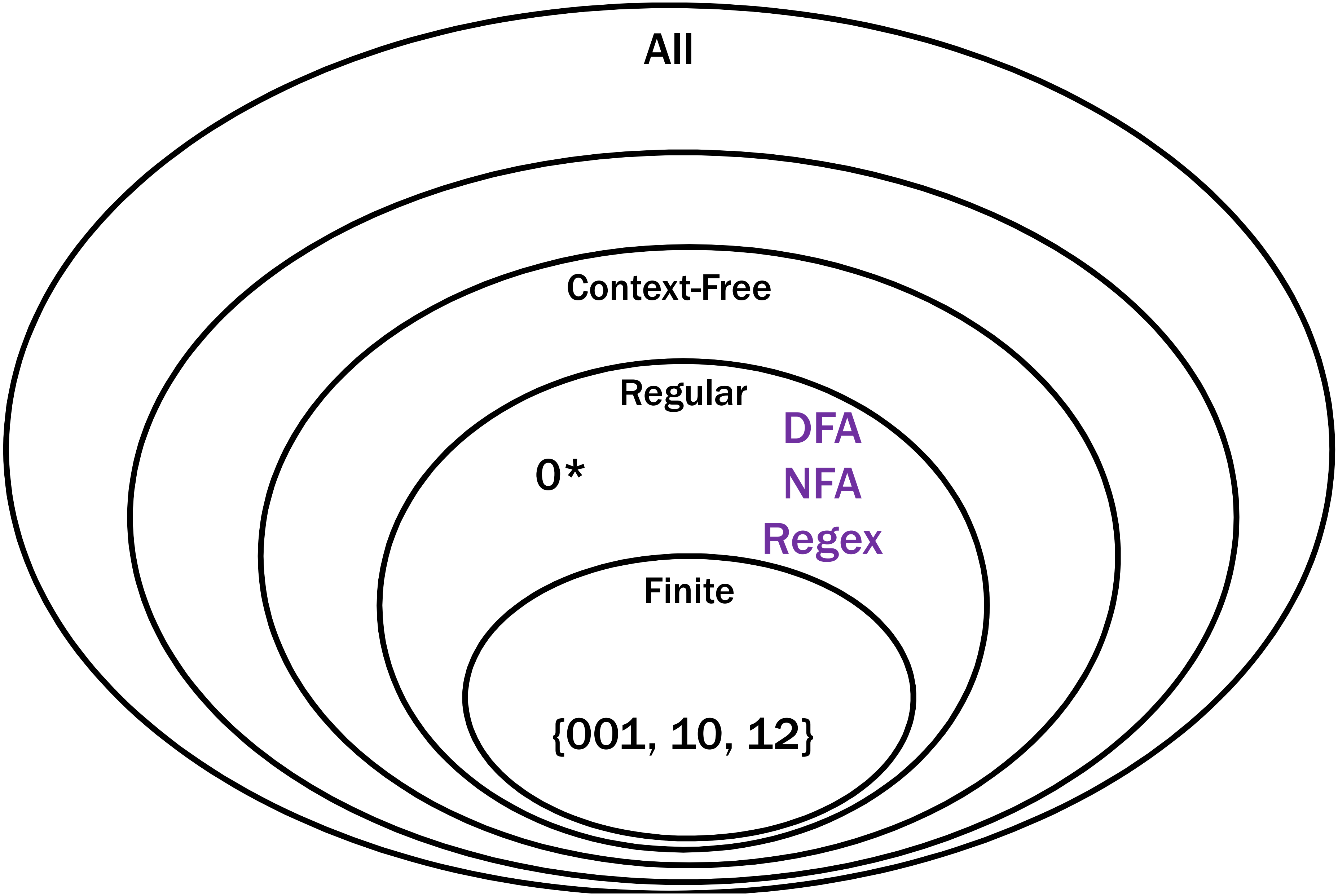
- See Extra Credit problem on HW8 for some ideas of how to do it.

$O(m^2 + n)$.

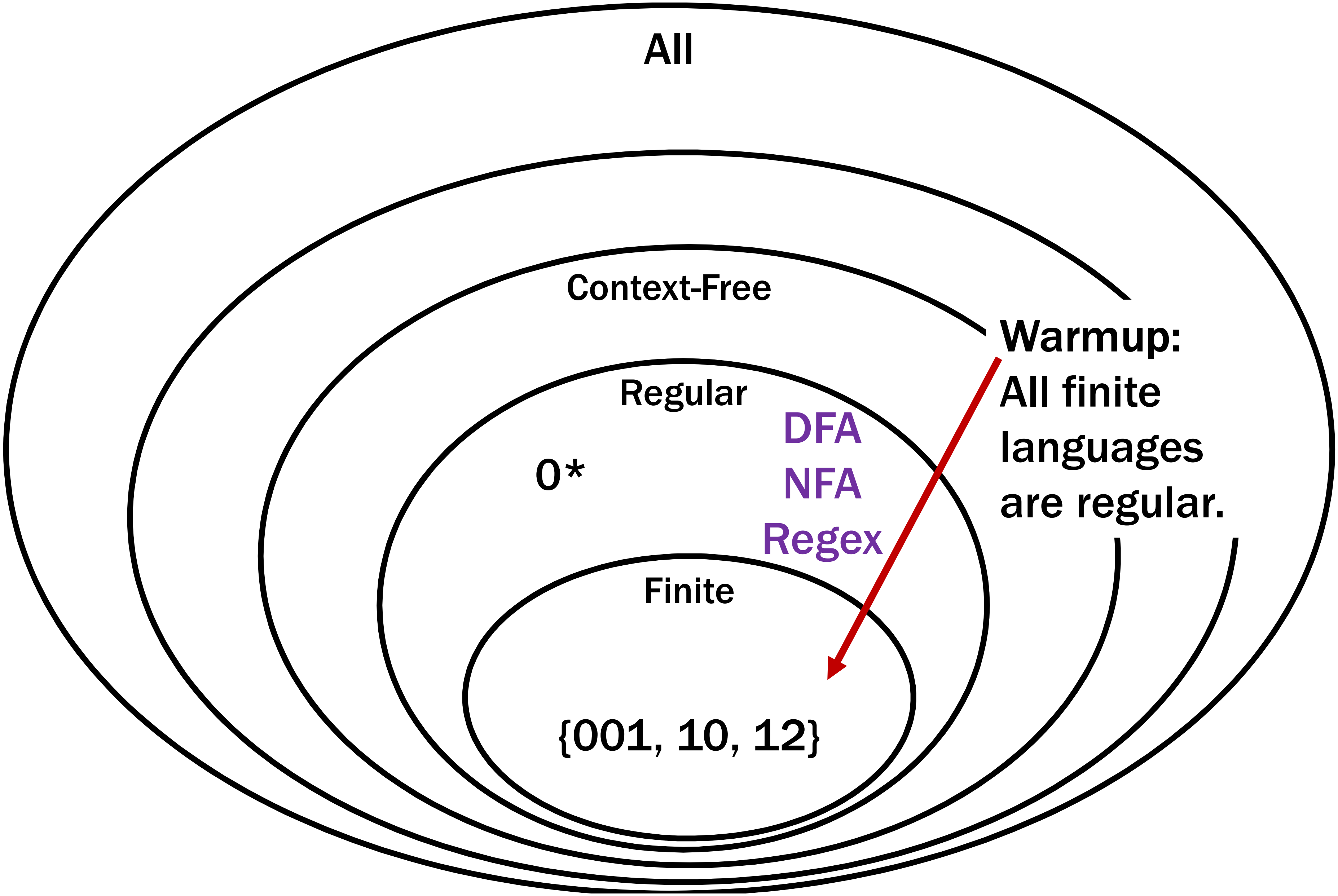
What languages have DFAs? CFGs?

All of them?

Languages and Representations!



Languages and Representations!



DFAs Recognize Any Finite Language

$B = \{x_1, x_2, \ldots, x_n\}$.

X_i is RE that recognizes $\{x_i\}$.

Unions are REs

→ $x_1 \cup x_2 \cup \ldots \cup x_n \in RE$

→ convert to NFA.

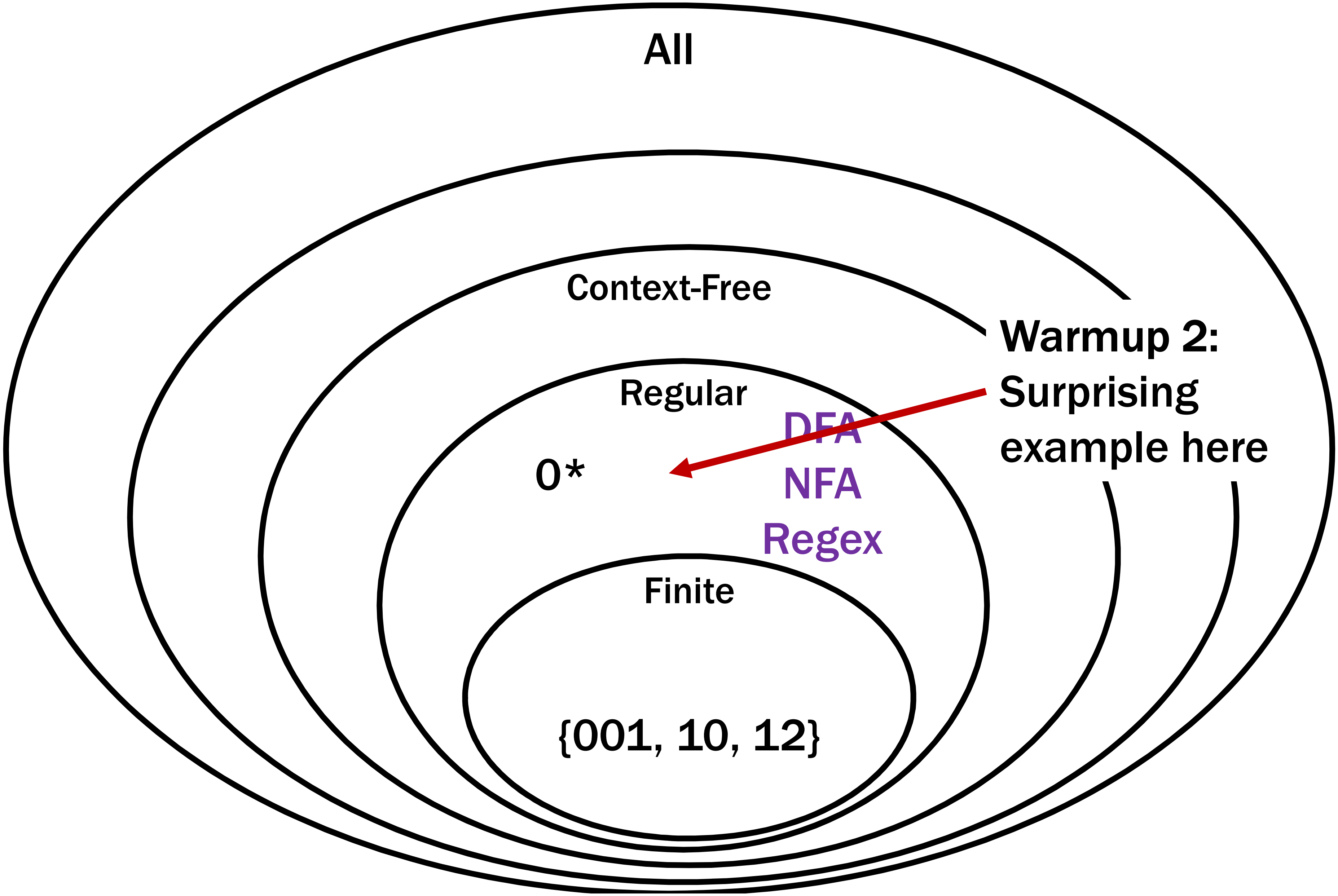
→ convert to DFA.

DFAs Recognize Any Finite Language

Construct a DFA for each string in the language.

Then, put them together using the union construction.

Languages and Machines!



An Interesting Infinite Regular Language

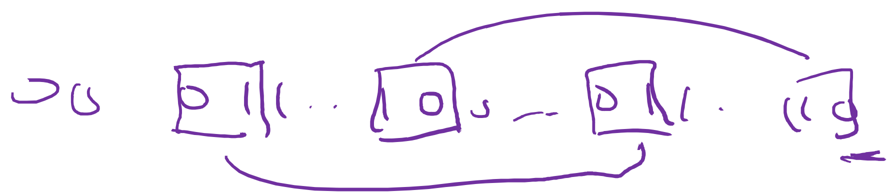
L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

L is infinite.

0, 00, 000, …

L is regular. How could this be?

(It seems to be comparing counts and counting seems hard for DFAs.)



An Interesting Infinite Regular Language

L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

L is infinite.

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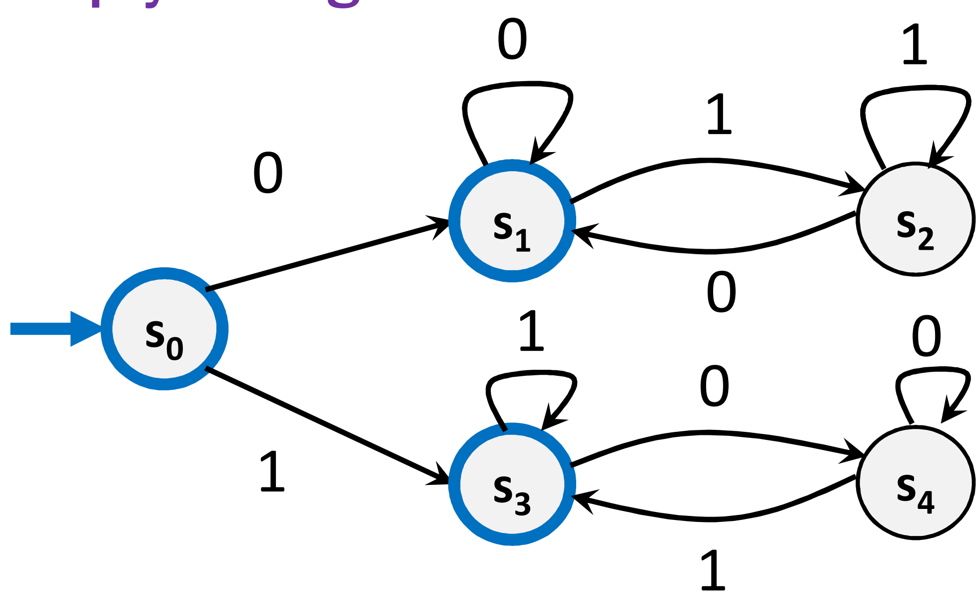
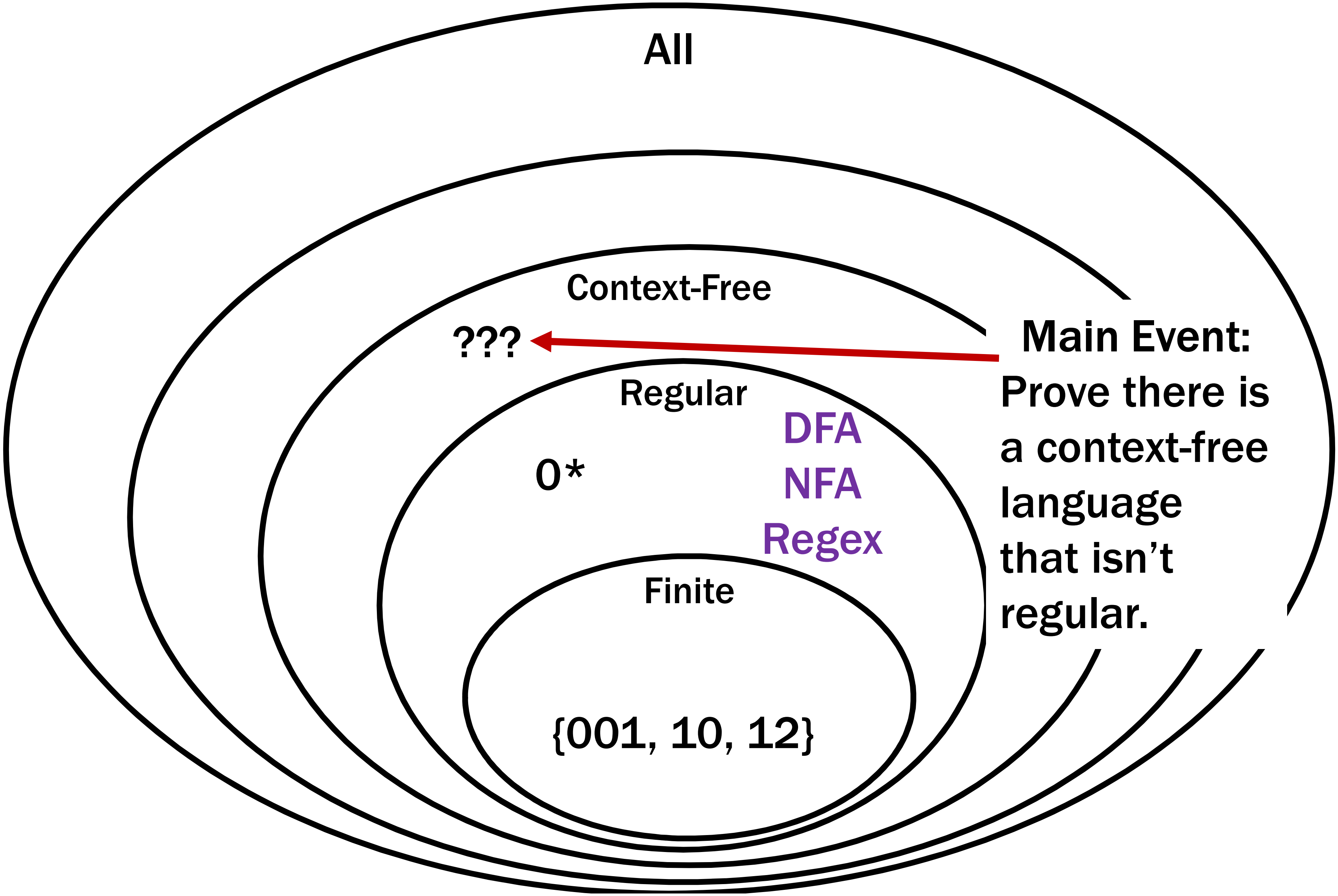
L is regular. How could this be? It is just the set of binary strings that are empty or begin and end with the same character!



Languages and Representations!



The language of “Binary Palindromes” is Context-Free

$$S \rightarrow \varepsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$$

S → 0S0

→ 01S10

⋮

0

Is the language of “Binary Palindromes” Regular ?

Is the language of “Binary Palindromes” Regular ?

Intuition (NOT A PROOF!):

Q: What would a DFA need to keep track of to decide the language?

A: It would need to keep track of the *first half* of the input in order to check the *second half* against it

…but there are an infinite # of possible first halves and we only have finitely many states.

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that it's possible.
- Therefore, some DFA (call it M) exists that recognizes B

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– Our goal is to show that M must be "confused"… we want to show that it "does the wrong thing".

How can a DFA be "wrong"?

– when it accepts or rejects a string it shouldn't.

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P be "M recognizes B"

$P \wedge \neg P \equiv F$

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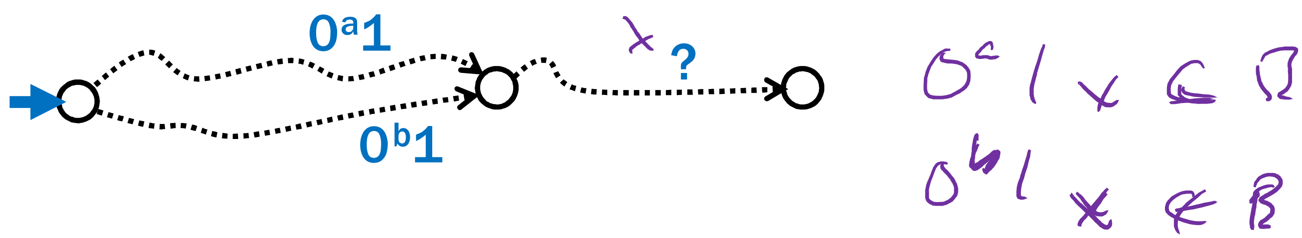
- We want to show: M accepts or rejects a string it shouldn't.

Key Idea 1: If two strings "collide" at any point, a DFA can no longer distinguish between them!



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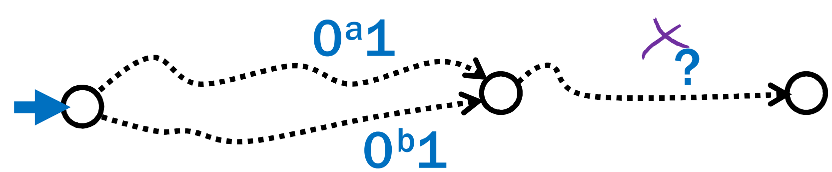

Key Idea 2: Our machine M has a finite number of states which means if we have infinitely many strings, two of them must collide!

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The general proof strategy is:

- Assume (for contradiction) that it's possible.
- Therefore, some DFA (call it M) exists that recognizes B
- We want to show: M accepts or rejects a string it shouldn't.

We choose an **INFINITE** set S of "partial strings" (which we intend to complete later). It is imperative that for *every* ***pair*** of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.

x

1______

01______

001______

0001______

00001______

....

for every pair 0^a1, 0^b1

$\exists x$ s.t. $0^a1x \in B$

$0^b1x \notin B$.

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, recognizes B.

We show M accepts or rejects a string it shouldn't.

Consider S={1, 01, 001, 0001, 00001, ...} = $\{0^n1 : n \geq 0\}$.

Key Idea 2: Our machine has a finite number of states which means if we have infinitely many strings, two of them must collide!

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, recognizes B.

We show M accepts or rejects a string it shouldn't.

Consider $S = \{0^n1 : n \geq 0\}$.

Since there are finitely many states in M and infinitely many strings in S, there exist strings $0^a1 \in S$ and $0^b1 \in S$ with $a \neq b$ that end in the same state of M.

SUPER IMPORTANT POINT: You do not get to choose what a and b are. Remember, we've just proven they exist…we have to take the ones we're given!

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, accepts B.

We show M accepts or rejects a string it shouldn't.

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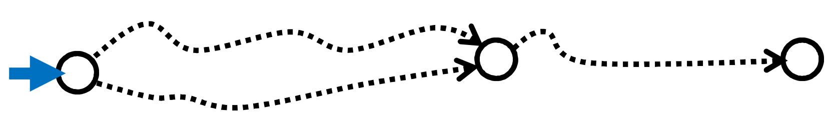
$x = 0^a$

$0^a1\,0^a \in B$

Now, consider appending 0^a *to both strings.*

$0^b1\,0^a \notin B$

Key Idea 1: If two strings "collide" at any point, a DFA can no longer distinguish between them!



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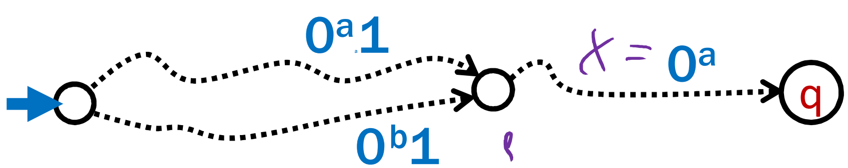

Then, since 0^a1 and 0^b1 end in the same state, 0^a10^a and 0^b10^a also end in the same state, call it q. But then M must make a mistake: q needs to be an accept state since $0^a10^a \in$ B, but then M would accept $0^b10^a \notin$ B which is an error.

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Suppose for contradiction that some DFA, M, recognizes B.

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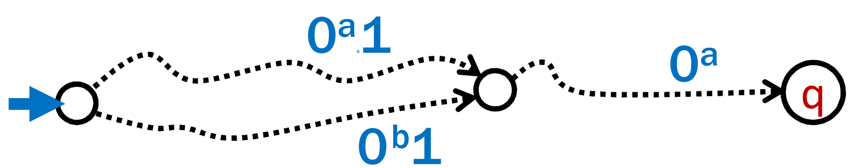

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This is a contradiction, since we assumed that M recognizes B. Hence, our assumption that such an M exists was false – there is no DFA that recognizes B. ■

Showing that a Language L is not regular

1. "Suppose for contradiction that some DFA M recognizes L."
2. Consider an INFINITE set S of "partial strings" (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.
3. "Since S is infinite and M has finitely many states, there must be two strings s_a and s_b in S for $s_a \neq s_b$ that end up at the same state of M."
4. Consider appending the (correct) completion t to each of the two strings.
5. "Since s_a and s_b both end up at the same state of M, and we appended the same string t, both $s_a t$ and $s_b t$ end at the same state q of M. Since $s_a t \in L$ and $s_b t \notin L$, M does not recognize L."
6. "Since M was arbitrary, no DFA recognizes L."

Prove $A = \{0^n 1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, **M**, recognizes **A**.

Let **S** = $\{0^n \mid n \geq 0\}$

Prove **A** = $\{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let S = $\{0^n : n \geq 0\}$**. Since S is infinite and M has finitely many states, there must be two strings,** 0^a **and** 0^b **for some** $a \neq b$ **that end in the same state in M.**

Prove $A = \{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, **M**, recognizes **A**.

Let $S = \{0^n : n \geq 0\}$. Since **S** is infinite and **M** has finitely many states, there must be two strings, 0^a and 0^b for some $a \neq b$ that end in the same state in **M**.

Consider appending 1^a to both strings.

$0^a 1^a \in A$

$0^b 1^a \notin A$

Prove A = $\{0^n1^n : n \geq 0\}$ is not regular

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Note that $0^a1^a \in$ A, but $0^b1^a \notin$ A since $a \neq b$. But they both end up in the same state of M, call it q. Since $0^a1^a \in$ A, state q must be an accept state but then M would incorrectly accept $0^b1^a \notin$ A so M does not recognize A.

Since M was arbitrary, no DFA recognizes A.

Prove P = {balanced parentheses} **is not regular**

Suppose for contradiction that some DFA, M, accepts P.

Let S =

Prove **P** = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, recognizes P.

Let $S = \{ (^n : n \geq 0\}$. **Since S is infinite and M has finitely many states, there must be two strings,** $(^a$ **and** $(^b$ **for some** $a \neq b$ **that end in the same state in M.**

$(^a)^a \in P$

$(^b)^a \notin P$

Prove P = {balanced parentheses} **is not regular**

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Let S = $\{ (^n : n \geq 0\}$**. Since S is infinite and M has finitely many states, there must be two strings,** $(^a$ **and** $(^b$ **for some** $a \neq b$ **that end in the same state in M.**

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Prove P = {balanced parentheses} is not regular

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Note that $(^a)^a \in P$, but $(^b)^a \notin P$ since $a \neq b$. But they both end up in the same state of M, call it q. Since $(^a)^a \in P$, state q must be an accept state but then M would incorrectly accept $(^b)^a \notin P$ so M does not recognize P.

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Fact: This method is optimal

- Suppose that for a language L, the set S is a *largest* set of "partial strings" with the property that for every pair $s_a \neq s_b \in S$, there is some string t such that one of $s_a t$, $s_b t$ is in L but the other isn't.
- If S is infinite then L is not regular
- If S is finite then the minimal DFA for L has precisely $|S|$ states, one reached by each member of S.

$|S|$

BTW: There is another method commonly used to prove languages not regular called the Pumping Lemma that we won't use in this course. Note that it doesn't always work.