Lecture 24: NFAs, Regular expressions, and NFA→DFA
Last time: Nondeterministic Finite Automata (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol—can have 0 or >1
  - Also can have edges labeled by empty string $\varepsilon$
- **Defn:** $x$ is in the language recognized by an NFA if and only if $x$ labels a path from the start state to some final state
Last time: Three ways of thinking about NFAs

• **Outside observer:** Is there a path labeled by $x$ from the start state to some final state?

• **Perfect guesser:** The NFA has input $x$ and whenever there is a choice of what to do it magically guesses a good one (if one exists)

• **Parallel exploration:** The NFA computation runs all possible computations on $x$ step-by-step at the same time in parallel
Last time: Compare with the smallest DFA
Last time: Parallel Exploration view of an NFA

Input string 0101100
Theorem: For any set of strings (language) $A$ described by a regular expression, there is an NFA that recognizes $A$.

Proof idea: Structural induction based on the recursive definition of regular expressions...
Regular Expressions over $\Sigma$

- **Basis:**
  - $\emptyset$, $\varepsilon$ are regular expressions
  - $a$ is a regular expression for any $a \in \Sigma$

- **Recursive step:**
  - If $A$ and $B$ are regular expressions then so are:
    - $(A \cup B)$
    - $(AB)$
    - $A^*$
Base Case

• Case $\emptyset$:

• Case $\varepsilon$:

• Case $a$:
Base Case

- Case $\emptyset$:

- Case $\varepsilon$:

- Case $a$:
Inductive Hypothesis

• Suppose that for some regular expressions $A$ and $B$ there exist NFAs $N_A$ and $N_B$ such that $N_A$ recognizes the language given by $A$ and $N_B$ recognizes the language given by $B$
Inductive Step

Case \((A \cup B)\):
Inductive Step

Case \((A \cup B)\):
Inductive Step

Case (AB):
Inductive Step

Case (AB):
Inductive Step

Case A*
Inductive Step

Case A*

![Diagram](image-url)
Build an NFA for \((01 \cup 1)^*0\)
Solution

\((01 \cup 1)^*0\)
NFAs and DFAs

Every DFA is an NFA
  - DFAs have requirements that NFAs don’t have

Can NFAs recognize more languages?
NFAs and DFAs

Every DFA is an NFA
– DFAs have requirements that NFAs don’t have

Can NFAs recognize more languages? No!

Theorem: For every NFA there is a DFA that recognizes exactly the same language
Three ways of thinking about NFAs

- Outside observer: Is there a path labeled by $x$ from the start state to some final state?

- Perfect guesser: The NFA has input $x$ and whenever there is a choice of what to do it magically guesses a good one (if one exists)

- Parallel exploration: The NFA computation runs all possible computations on $x$ step-by-step at the same time in parallel
Conversion of NFAs to a DFAs

• Proof Idea:
  – The DFA keeps track of ALL the states that the part of the input string read so far can reach in the NFA
  – There will be one state in the DFA for each subset of states of the NFA that can be reached by some string
Parallel Exploration view of an NFA

Input string 0101100

The diagram shows the NFA's state transitions and the input string parsed through it.
Conversion of NFAs to a DFAs

New start state for DFA

– The set of all states reachable from the start state of the NFA using only edges labeled $\varepsilon$
Conversion of NFAs to a DFAs

For each state of the DFA corresponding to a set $S$ of states of the NFA and each symbol $s$

- Add an edge labeled $s$ to state corresponding to $T$, the set of states of the NFA reached by
  - starting from some state in $S$, then
  - following one edge labeled by $s$, and
  - then following some number of edges labeled by $\varepsilon$
- $T$ will be $\emptyset$ if no edges from $S$ labeled $s$ exist
Final states for the DFA

- All states whose set contain some final state of the NFA
Example: NFA to DFA
Example: NFA to DFA
Example: NFA to DFA

NFA

DFA
Example: NFA to DFA

NFA

DFA
Example: NFA to DFA
Example: NFA to DFA
Example: NFA to DFA
Example: NFA to DFA
Exponential Blow-up in Simulating Nondeterminism

• In general the DFA might need a state for every subset of states of the NFA
  – Power set of the set of states of the NFA
  – $n$-state NFA yields DFA with at most $2^n$ states
  – We saw an example where roughly $2^n$ is necessary
    “Is the $n^{th}$ char from the end a 1?”

• The famous “P=NP?” question asks whether a similar blow-up is always necessary to get rid of nondeterminism for polynomial-time algorithms
DFAs ≡ NFAs ≡ Regular expressions

We have shown how to build an optimal DFA for every regular expression
  – Build NFA
  – Convert NFA to DFA using subset construction
  – Minimize resulting DFA

**Theorem:** A language is recognized by a DFA (or NFA) if and only if it has a regular expression

You need to know this fact but we won’t ask you anything about the “only if” direction from DFA/NFA to regular expression. For fun, we sketch the idea.
Generalized NFAs

• Like NFAs but allow
  – Parallel edges
  – Regular Expressions as edge labels
    NFAs already have edges labeled $\varepsilon$ or $a$
• An edge labeled by $A$ can be followed by reading a string of input chars that is in the language represented by $A$
• Defn: A string $x$ is accepted iff there is a path from start to final state labeled by a regular expression whose language contains $x$
Starting from an NFA

Add new start state and final state

Then eliminate original states one by one, keeping the same language, until it looks like:

Final regular expression will be A
Only two simplification rules

- **Rule 1:** For any two states $q_1$ and $q_2$ with parallel edges (possibly $q_1=q_2$), replace

  \[ q_1 \xrightarrow{A} q_2 \text{ by } q_1 \xrightarrow{A \cup B} q_2 \]

- **Rule 2:** Eliminate non-start/final state $q_3$ by replacing all

  \[ q_1 \xrightarrow{A} q_3 \xrightarrow{C} q_2 \text{ by } q_1 \xrightarrow{A B^* C} q_2 \]

for every pair of states $q_1, q_2$ (even if $q_1=q_2$)
Converting an NFA to a regular expression

Consider the DFA for the mod 3 sum

- Accept strings from \( \{0,1,2\}^* \) where the digits mod 3 sum of the digits is 0
Splicing out a state $t_1$

Regular expressions to add to edges

- $t_0 \rightarrow t_1 \rightarrow t_0 : 10^*2$
- $t_0 \rightarrow t_1 \rightarrow t_2 : 10^*1$
- $t_2 \rightarrow t_1 \rightarrow t_0 : 20^*2$
- $t_2 \rightarrow t_1 \rightarrow t_2 : 20^*1$
Splicing out a state $t_1$

Regular expressions to add to edges

$t_0 \to t_1 \to t_0 : 10^*2$
$t_0 \to t_1 \to t_2 : 10^*1$
$t_2 \to t_1 \to t_0 : 20^*2$
$t_2 \to t_1 \to t_2 : 20^*1$
Splicing out state $t_2$ (and then $t_0$)

$R_1: \ 0 \cup 10*2$
$R_2: \ 2 \cup 10*1$
$R_3: \ 1 \cup 20*2$
$R_4: \ 0 \cup 20*1$

$R_5: \ R_1 \cup R_2 R_4 * R_3$

Final regular expression: $R_5^* = (0 \cup 10*2 \cup (2 \cup 10*1)(0 \cup 20*1)*(1 \cup 20*2))^*$