CSE 311: Foundations of Computing

Lecture 18: Structural Induction, Regular expressions

Recursive Definitions of Sets: General Form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from the basis step and a finite number of recursive steps

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Strings

- An *alphabet* Σ is any finite set of characters
- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma$ (ε **is the empty string w/ no chars)**
  - **Recursive:** if $\underline{w \in \Sigma^*}$, $\underline{a \in \Sigma}$, then $\underline{wa} \in \Sigma^*$

Σ^* is the set of all strings $\underline{a_1 a_2 \cdots a_n}$ for some $n \geq 0$ and $a_i \in \Sigma$ $(\forall i)$

Functions on Recursively Defined Sets (on Σ^*)

Length:

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$

Number of c's in a string:

$\#_c(\varepsilon) = 0$

$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

Claim: len(x•y) = len(x) + len(y) **for all** x,y ∈ Σ*

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x ∈ Σ* "**.**
We prove P(y) **for all** y ∈ Σ* **by structural induction.**

$\forall x \in S$

Base Case ($y = \varepsilon$): Let $x \in \Sigma^*$ be arbitrary.
(Prove $P(\varepsilon)$)

$\text{len}(x \cdot \varepsilon) = \text{len}(x)$ defn of $\circ$

$= \text{len}(x) + 0$ algebra

$= \text{len}(x) + \text{len}(\varepsilon)$ defn of len

Since x was arbitrary, this proves $P(\varepsilon)$.

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$

Claim: len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x $\in \Sigma^*$" .
We prove P(y) **for all** y $\in \Sigma^*$ **by structural induction.**

Base Case: y= ε. **For any** x $\in \Sigma^*$, len(x• ε) = len(x) = len(x) + len(ε) **since** len(ε)=0. **Therefore** P(ε) **is true**

Claim: len(x•y) = len(x) + len(y) for all x,y $\in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x $\in \Sigma^*$" .
We prove P(y) **for all** y $\in \Sigma^*$ **by structural induction.**

Base Case: y= ε. **For any** x $\in \Sigma^*$, len(x• ε) = len(x) = len(x) + len(ε)
since len(ε)=0. **Therefore** P(ε) **is true**

Inductive Hypothesis: Assume that P(w) **is true for some arbitrary** w $\in \Sigma^*$

y = wa

Inductive Step: $\boxed{\text{Goal: Show that P(wa) is true for every a} \in \Sigma}$

Let $x \in \Sigma^*$ be arbitrary.

$$\begin{aligned} len(x\cdot(wa)) &= len((x\cdot w)a) && \text{defn of } \cdot \\ &= len(x\cdot w)+1 && \text{defn of len} \\ &= len(x)+len(w)+1 && \text{I.H.} \\ &= len(x)+len(wa) && \text{defn of len} \end{aligned}$$

Since x was arbitrary, thus prove P(wa).

Claim: len($x \bullet y$) = len(x) + len(y) **for all** $x,y \in \Sigma^*$

Let P(y) **be** "len($x \bullet y$) = len(x) + len(y) for all $x \in \Sigma^*$" .
We prove P(y) **for all** $y \in \Sigma^*$ **by structural induction.**

Base Case: $y = \varepsilon$. **For any** $x \in \Sigma^*$, len($x \bullet \varepsilon$) = len(x) = len(x) + len(ε) **since** len(ε)=0. **Therefore** P(ε) **is true**

Inductive Hypothesis: Assume that P(w) **is true for some arbitrary** $w \in \Sigma^*$

Inductive Step: $\boxed{\textbf{Goal: Show that } P(wa) \textbf{ is true for every } a \in \Sigma}$

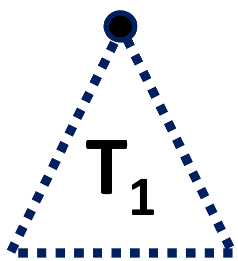
Let $a \in \Sigma$. **Let** $x \in \Sigma^*$. **Then** len($x \bullet wa$) = len(($x \bullet w$)a) **by defn of** $\bullet$
= len($x \bullet w$)+1 **by defn of len**
= len(x)+len(w)+1 **by I.H.**
= len(x)+len(wa) **by defn of len**

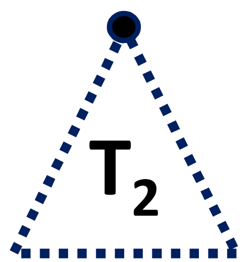
Therefore len($x \bullet wa$)= len(x)+len(wa) **for all** $x \in \Sigma^*$, **so** P(wa) **is true.**

So, by induction len($x \bullet y$) = len(x) + len(y) **for all** $x,y \in \Sigma^*$

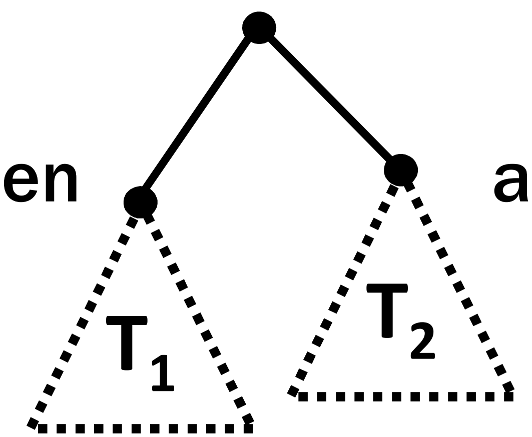
Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If 

and 

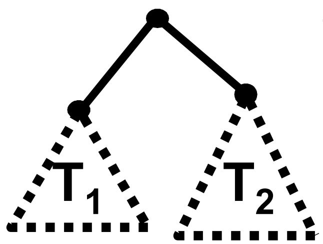
are rooted binary trees,

then 

also is a rooted binary tree.

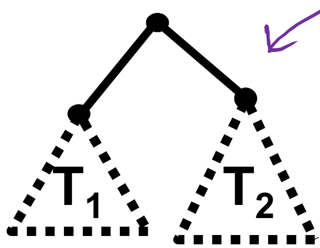
Defining Functions on Rooted Binary Trees

- size(•) = 1

- size (

) = 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)

- height(•) = 0

- height (

)=1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

Claim: For every rooted binary tree T, size(T) $\leq 2^{\text{height}(T)+1} - 1$

1. Let P(T) be "size(T) $\leq 2^{\text{height}(T)+1} - 1$". We prove P(T) for all rooted binary trees T by structural induction.

Base Case: $(\cdot)$:

$\text{size}(\cdot) = 1$

$\text{height}(\cdot) = 0$.

Let

so $2^{\text{ht}(\cdot)+1} - 1 = 2^{0+1} - 1$

$= 2^1 - 1 = 2 - 1 = 1$

Since, $1 \leq 1$, $P(\cdot)$ is true.

Claim: For every rooted binary tree $\mathbf{T}$, $\text{size}(\mathbf{T}) \leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

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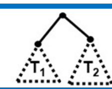
1. **Let** $P(\mathbf{T})$ **be** "$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$". **We prove** $P(\mathbf{T})$ **for all rooted binary trees T by structural induction.**
2. **Base Case:** $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ **and** $1=2^1-1=2^{0+1}-1$ **so** $P(\bullet)$ **is true.**

Claim: For every rooted binary tree T, size(T) $\le 2^{\text{height}(T)+1} - 1$

1. Let P(T) be "size(T) $\le 2^{\text{height}(T)+1}-1$". We prove P(T) for all rooted binary trees T by structural induction.
2. **Base Case:** size(•)=1, height(•)=0 and $1=2^1-1=2^{0+1}-1$ so P(•) is true.
3. **Inductive Hypothesis:** Suppose that P(T_1) and P(T_2) are true for some rooted binary trees T_1 and T_2.
4. **Inductive Step:** Goal: Prove P().

$$\text{size}(T) = 1 + \underline{\text{size}(T_1)} + \underline{\text{size}(T_2)} \qquad \text{defn of size}$$

$$\le 1 + 2^{ht(T_1)+1} - 1 + \text{size}(T_2)$$

$$\le \cancel{1} + 2^{ht(T_1)+1} \cancel{-1} + 2^{ht(T_2)+1} - 1 \qquad \text{I.H.}$$

$$= 2\left(2^{ht(T_1)} + 2^{ht(T_2)}\right) - 1 \qquad \text{algebra}$$

$$\le 2\left(2 \cdot 2^{\max\{ht(T_1), ht(T_2)\}}\right) - 1$$

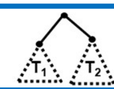
$$= 2\left(2^{\max\{ht(T_1), ht(T_2)\}+1}\right) - 1 \qquad \text{algebra}$$

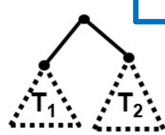
$$= 2 \cdot 2^{ht(T)} - 1 = 2^{ht(T)+1} - 1.$$

$$ht(T) = \max\{ht(T_1), ht(T_2)\} + 1.$$

Claim: For every rooted binary tree $\mathbf{T}$, $\text{size}(\mathbf{T}) \le 2^{\text{height}(\mathbf{T})+1} - 1$

1. **Let** $P(\mathbf{T})$ **be** "$\text{size}(T) \le 2^{\text{height}(T)+1}-1$". **We prove** $P(\mathbf{T})$ **for all rooted binary trees T by structural induction.**
2. **Base Case:** $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ **and** $1=2^1-1=2^{0+1}-1$ **so** $P(\bullet)$ **is true.**
3. **Inductive Hypothesis: Suppose that** $P(\mathbf{T}_1)$ **and** $P(\mathbf{T}_2)$ **are true for some rooted binary trees** $\mathbf{T}_1$ **and** $\mathbf{T}_2$**.**
4. **Inductive Step:**

Goal: Prove $P($  $)$**.**

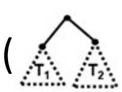
By defn, size() $=1+\text{size}(\mathbf{T}_1)+\text{size}(\mathbf{T}_2)$

$\le 1+2^{\text{height}(\mathbf{T}_1)+1}-1+2^{\text{height}(\mathbf{T}_2)+1}-1$

by IH for $\mathbf{T}_1$ and $\mathbf{T}_2$

$= 2^{\text{height}(\mathbf{T}_1)+1}+2^{\text{height}(\mathbf{T}_2)+1}-1$

$\le 2(2^{\max(\text{height}(\mathbf{T}_1),\text{height}(\mathbf{T}_2))+1})-1$

$= 2(2^{\text{height}($  $)}) - 1 = 2^{\text{height}($  $)+1} - 1$

which is what we wanted to show.

5. **So, the** $P(T)$ **is true for all rooted bin. trees by structural induction.**

Languages: Sets of Strings

- Sets of strings that satisfy special properties are called *languages*. Examples:
  - English sentences
  - Syntactically correct Java/C/C++ programs
  - $\Sigma^* =$ All strings over alphabet Σ
  - Palindromes over Σ
  - Binary strings that don't have a 0 after a 1
  - Legal variable names. keywords in Java/C/C++
  - Binary strings with an equal # of 0's and 1's

Regular Expressions

Regular expressions over Σ

- **Basis:**
  - $\varnothing$, ε are regular expressions
  - $\boldsymbol{a}$ is a regular expression for any $a \in \Sigma$
- **Recursive step:**
  - If **A** and **B** are regular expressions then so are:
    - $(\mathbf{A} \cup \mathbf{B})$
    - $(\mathbf{AB})$
    - $\mathbf{A}^*$

Each Regular Expression is a "pattern"

ε matches the **empty string**

a matches the one character string *a*

(A ∪ B) matches all strings that either **A** matches
or **B** matches (or both)

(AB) matches all strings that have a first part that
A matches followed by a second part that **B**
matches

Σ^*

A* matches all strings that have any number of
strings (even 0) that **A** matches, one after
another

ε
x_1
$x_1 x_2$
$x_1 x_2 x_3$.

x_i match A.

Examples

001*

{ 00, 001, 0011, 00111, ... }

0*1*

00...011...1

Examples

001*

{00, 001, 0011, 00111, ...}

0*1*

Any number of 0’s followed by any number of 1’s

Examples

$(0 \cup 1)\, 0\, (0 \cup 1)\, 0$

{0000, 0010, 1000, 1010}.

$(0^*1^*)^*$

all strings of 0's & 1's.

$(0 \cup 1)^*$

Examples

$(0 \cup 1)\ 0\ (0 \cup 1)\ 0$

{0000, 0010, 1000, 1010}

$(0^*1^*)^*$

All binary strings

Examples

$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$

$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$

Examples

$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$

Binary strings that contain “0110”

$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$

Binary strings that begin with pairs of characters followed by “01010” or “10001”

Regular Expressions in Practice

- Used to define the "tokens": e.g., legal variable names, keywords in programming languages and compilers
- Used in **`grep`**, a program that does pattern matching searches in UNIX/LINUX
- Pattern matching using regular expressions is an essential feature of PHP
- We can use regular expressions in programs to process strings!

Regular Expressions in Java

- Pattern p = Pattern.compile("a*b");
- Matcher m = p.matcher("aaaaab");
- boolean b = m.matches();

`[01]` a 0 or a 1	`^` start of string	`$` end of string	
`[0-9]` any single digit	`\.` period	`\,` comma	`\-` minus
`.` any single character			

ab	a followed by b	(**AB**)
(a | b)	a or b	(**A** $\cup$ **B**)
a?	zero or one of a	(**A** $\cup$ ε)
a*	zero or more of a	**A***
a+	one or more of a	**AA***

a{n,m}

- e.g. `^[\-+]?[0-9]*(\.|\,)?[0-9]+$`

  General form of decimal number e.g. 9.12 or -9,8 (Europe)