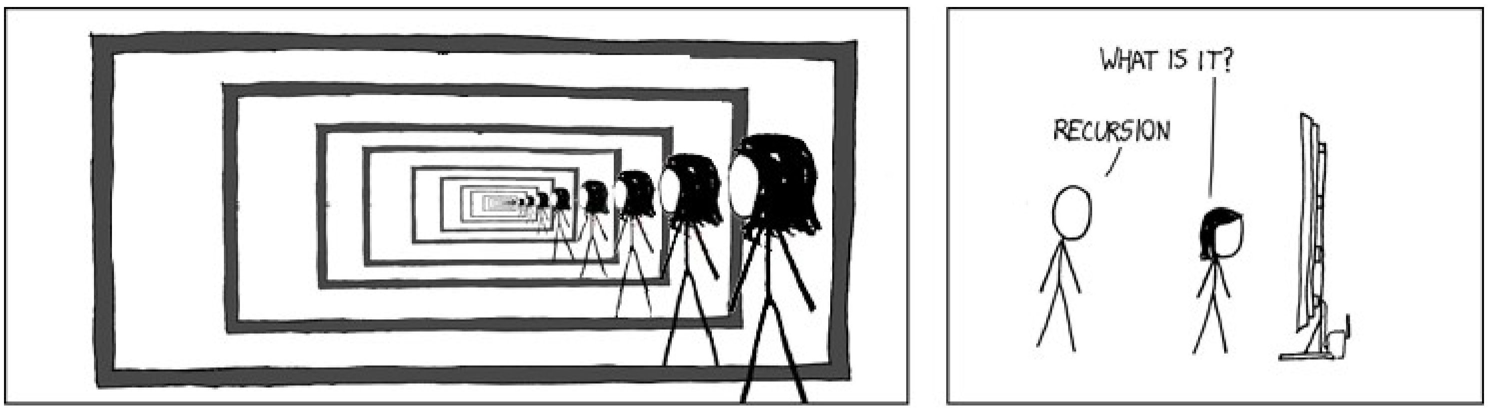
CSE 311: Foundations of Computing

Lecture 17: Recursively Defined Sets & Structural Induction



Recursive Definition of Sets

Recursive definition of set S

- **Basis Step:** $0 \in S$
- **Recursive Step: If** $x \in S$**, then** $x + 2 \in S$
- **Exclusion Rule: Every element in** S **follows from the basis step and a finite number of recursive steps.**

$2n$ for $n \geq 0$

We need the exclusion rule because otherwise $S=\mathbb{N}$ **would satisfy the other two parts. However, we won't always write it down on these slides.**

Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$

Recursive: **If** $x,y \in S$, **then** $x+y \in S$

6, 12=6+6, 18=12+6,

– 6n ∀n≥1

6n + 15m

15m ∀n≥1

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$

Recursive: **If** $[x, y, z] \in S$, **then** $[\alpha x, \alpha y, \alpha z] \in S$ **for any** $\alpha \in \mathbb{R}$ scaling.

If $[x_1, y_1, z_1] \in S$ **and** $[x_2, y_2, z_2] \in S$, **then**

$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$. element-wise addition

(a, a, 0) ∀a∈ℝ.

Number of form 3^n **for** $n \geq 0$: (0, b, b) ∀b∈ℝ.

(a, a+b, b) ∀a,b∈ℝ.

Basis: 1∈S

If x∈S,

3x∈S.

Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$

Recursive: If $x, y \in S$**, then** $x+y \in S$

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$

Recursive: If $[x, y, z] \in S$**, then** $[\alpha x, \alpha y, \alpha z] \in S$ **for any** $\alpha \in \mathbb{R}$

If $[x_1, y_1, z_1] \in S$ **and** $[x_2, y_2, z_2] \in S$**, then**

$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$**.**

Number of form 3^n **for** $n \geq 0$**:**

Basis: $1 \in S$

Recursive: If $x \in S$**, then** $3x \in S$**.**

Recursive Definitions of Sets: General Form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from the basis step and a finite number of recursive steps

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma^*$ (ε is the empty string w/ no chars)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

$a_1 a_2 \cdots a_n \in \Sigma$

$\varepsilon, \quad \varepsilon a_1 = a_1, \quad a_1 a_2, \ldots$

$a_1 a_2 \ldots a_n$

Palindromes

Palindromes are strings that are the same backwards and forwards

$\varepsilon\varepsilon = \varepsilon$

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

$a_1 \cdot a_{n-1} a_n a_{n-1} \cdots a_1$

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$

$\boxed{a_1 a_2 \ldots a_n a_n \,.\, a_2 a_1}$

$2n$

$\varepsilon \quad a_n \varepsilon a_n = a_n a_n, \quad a_{n-1} a_n a_n a_{n-1}, \ldots$

All Binary Strings with no 1's before 0's

Basis: $\varepsilon \in S$ $\qquad \Sigma = \{0, 1\}$.

Recursive: If $x \in S$,

$0x \in S$ ←

and $x1 \in S$. ←

$\varepsilon, 0, \ldots, 0^n$

$, 0^m 1, \ldots, 0^n 1^m$

$0^n 1^m \quad m, n \geq 0$.

All Binary Strings with no 1's before 0's

Basis:

$\varepsilon \in S$

Recursive:

If $x \in S$, **then** $0x \in S$

If $x \in S$, **then** $x1 \in S$

Functions on Recursively Defined Sets (on Σ^*)

Length:

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

$\varepsilon \leftarrow$

$wa \leftarrow$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

$(abc)^R$

$((ab)c)^R = c(ab)^R$

$= c(ba^R)$

$= cb(\varepsilon a)^R$

$= cb(a\varepsilon^R)$

$= cba\varepsilon$

$= cba$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $w \in \Sigma^*$, $a \in \Sigma$

y

Number of c's in a string:

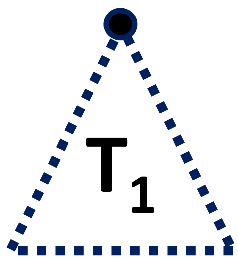
$\#_c(\varepsilon) = 0$

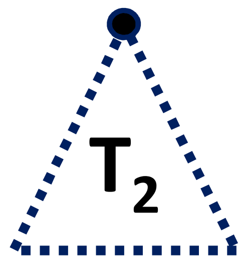
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

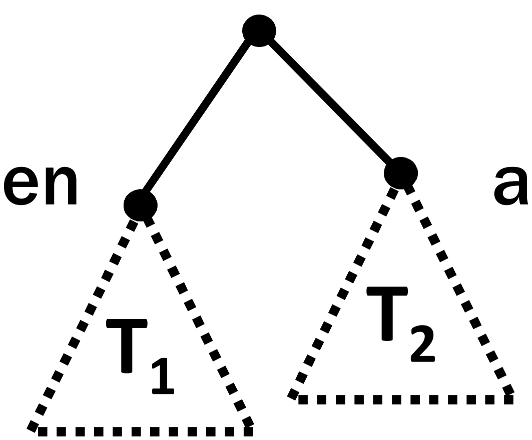
Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If 

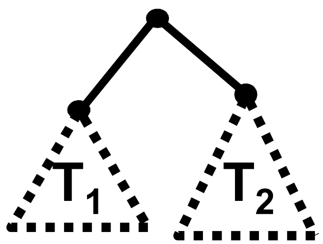
and 

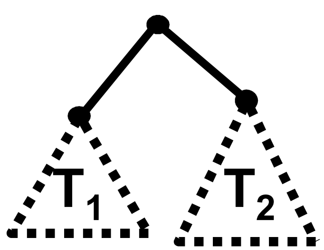
are rooted binary trees,

then 

also is a rooted binary tree.

Defining Functions on Rooted Binary Trees

- size(•) = 1
- size (

) = 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)
- height(•) = 0
- height (

)=1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

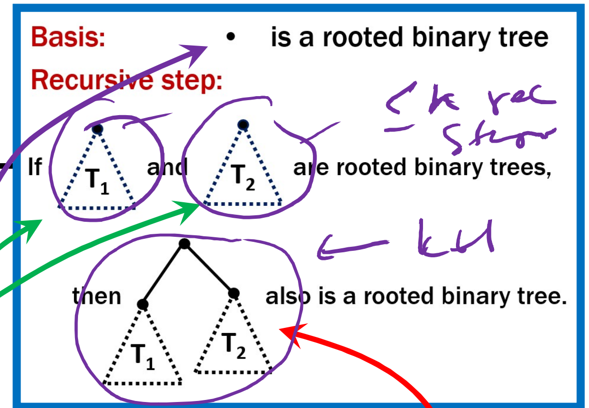
Base Case: **Show that $P(u)$ is true for all specific elements u of S mentioned in the** ***Basis step***

Inductive Hypothesis: **Assume that P is true for some arbitrary values of** ***each*** **of the existing named elements mentioned in the** ***Recursive step***

Inductive Step: **Prove that $P(w)$ holds for each of the new elements w constructed in the** ***Recursive step*** **using the named elements mentioned in the Inductive Hypothesis**

Conclude **that $\forall\, x \in S, P(x)$**

Structural Induction



How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Structural Induction vs. Ordinary Induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Define $Q(n)$ to be "for all $x \in S$ that can be constructed in at most n recursive steps, $P(x)$ is true."

$Q(0) \equiv$ true for basis elements.

Using Structural Induction

- Let S be given by…
  - Basis: $6 \in S$; $15 \in S$;
  - Recursive: if $x, y \in S$ then $x + y \in S$.

Claim: Every element of S is divisible by 3.

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$\forall x \in S$.

1. Let P(x) be "$3|x$". We prove that P(x) is true for all $x \in S$ by structural induction.
2. Base Case: $3|6$ and $3|15$ so P(6) and P(15) are true

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: Every element of S is divisible by 3.

1. Let P(x) be "3|x". We prove that P(x) is true for all x ∈ S by structural induction.
2. Base Case: 3|6 and 3|15 so P(6) and P(15) are true
3. Inductive Hypothesis: Suppose that P(x) and P(y) are true for some arbitrary x,y ∈ S
4. Inductive Step: $\boxed{\text{Goal: Show P(x+y)}}$

$3 \mid x$ means $x = 3m$ for some m.

$3 \mid y$ means $y = 3n$ for some n

$x+y = 3n + 3m = 3(n+m)$, which says

$3 \mid x+y$ P(x+y).

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$ ✓

Claim: Every element of S is divisible by 3.

1. Let $P(x)$ be "$3|x$". We prove that $P(x)$ is true for all $x \in S$ by structural induction.
2. Base Case: $3|6$ and $3|15$ so $P(6)$ and $P(15)$ are true
3. Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x,y \in S$
4. Inductive Step: $\boxed{\text{Goal: Show } P(x+y)}$

   Since $P(x)$ is true, $3|x$ and so $x=3m$ for some integer m and since $P(y)$ is true, $3|y$ and so $y=3n$ for some integer n. Therefore $x+y=3m+3n=3(m+n)$ and thus $3|(x+y)$.

   Hence $P(x+y)$ is true.
5. Therefore by induction $3|x$ for all $x \in S$.

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: len(x•y) = len(x) + len(y) **for all** x,y ∈ Σ*

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x ∈ Σ* ".
We prove P(y) **for all** y ∈ Σ* **by structural induction.**

Basis case: $\mathrm{len}(x \cdot \varepsilon) = \mathrm{len}(x)$ by defn of •

$= \mathrm{len}(x) + 0$

$= \mathrm{len}(x) + \mathrm{len}(\varepsilon)$ defn of len.

this proves $P(\varepsilon)$.

$\varepsilon \in \Sigma^*$

$wa \in \Sigma^*$.

$\mathrm{len}(\varepsilon) = 0$

$\mathrm{len}(wa) = \mathrm{len}(w) + 1$.

$x \cdot \varepsilon = x$ $\quad x \cdot (wa) = (x \cdot w)a$.

Claim: len(x•y) = len(x) + len(y) **for all** x,y ∈ Σ^*

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x ∈ Σ^*" .
We prove P(y) **for all** y ∈ Σ^* **by structural induction.**

Base Case: y= ε. **For any** x ∈ Σ^*, len(x• ε) = len(x) = len(x) + len(ε)
since len(ε)=0. **Therefore** P(ε) **is true**

Claim: len(x•y) = len(x) + len(y) for all x,y $\in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x $\in \Sigma^*$" .
We prove P(y) **for all** y $\in \Sigma^*$ **by structural induction.**

Base Case: y= ε. **For any** x $\in \Sigma^*$, len(x• ε) = len(x) = len(x) + len(ε) **since** len(ε)=0. **Therefore** P(ε) **is true**

y = wa.

Inductive Hypothesis: Assume that P(w) **is true for some arbitrary** w $\in \Sigma^*$

Inductive Step: $\boxed{\text{Goal: Show that } P(wa) \text{ is true for every } a \in \Sigma}$

Let $x \in \Sigma^*$ be arbitrary.

$len(x \cdot (wa)) = len((x \cdot w) a)$ defn of •

$= len(x \cdot w) + 1$ defn of len

$= len(x) + len(w) + 1$ by I.H.

$= len(x) + len(wa)$ defn of len

P(wa)

Claim: $\text{len}(x\bullet y) = \text{len}(x) + \text{len}(y)$ **for all** $x,y \in \Sigma^*$

Let $P(y)$ **be** "$\text{len}(x\bullet y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$".
We prove $P(y)$ **for all** $y \in \Sigma^*$ **by structural induction.**

Base Case: $y = \varepsilon$. **For any** $x \in \Sigma^*$, $\text{len}(x\bullet \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ **since** $\text{len}(\varepsilon)=0$. **Therefore** $P(\varepsilon)$ **is true**

Inductive Hypothesis: Assume that $P(w)$ **is true for some arbitrary** $w \in \Sigma^*$

Inductive Step: **Goal: Show that** $P(wa)$ **is true for every** $a \in \Sigma$

Let $a \in \Sigma$**. Let** $x \in \Sigma^*$. **Then** $\text{len}(x\bullet wa) = \text{len}((x\bullet w)a)$ **by defn of** $\bullet$
$= \text{len}(x\bullet w)+1$ **by defn of len**
$= \text{len}(x)+\text{len}(w)+1$ **by I.H.**
$= \text{len}(x)+\text{len}(wa)$ **by defn of len**

Therefore $\text{len}(x\bullet wa)= \text{len}(x)+\text{len}(wa)$ **for all** $x \in \Sigma^*$**, so** $P(wa)$ **is true.**

So, by induction $\text{len}(x\bullet y) = \text{len}(x) + \text{len}(y)$ **for all** $x,y \in \Sigma^*$

Claim: **For every rooted binary tree $\mathbf{T}$,** $\text{size}(\mathbf{T}) \leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

Claim: For every rooted binary tree **T**, size(**T**) $\leq 2^{\text{height}(\mathbf{T})+1} - 1$

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1. **Let** P(**T**) **be** "size(T) $\leq 2^{\text{height}(T)+1}-1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
2. **Base Case:** size(•)=1, height(•)=0 **and** $1=2^1-1=2^{0+1}-1$ **so** P(•) **is true.**

Claim: For every rooted binary tree T, size(T) $\leq 2^{\text{height}(\mathbf{T})+1} - 1$

1. **Let** P(**T**) **be** "size(T) $\leq 2^{\text{height}(T)+1}-1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
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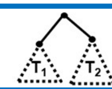
Goal: Prove P().

Claim: For every rooted binary tree T, $\text{size}(\mathbf{T}) \le 2^{\text{height}(\mathbf{T})+1} - 1$

1. **Let** $P(\mathbf{T})$ **be** "$\text{size}(T) \le 2^{\text{height}(T)+1}-1$". **We prove** $P(\mathbf{T})$ **for all rooted binary trees T by structural induction.**
2. **Base Case:** $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ **and** $1=2^1-1=2^{0+1}-1$ **so** $P(\bullet)$ **is true.**
3. **Inductive Hypothesis: Suppose that** $P(\mathbf{T_1})$ **and** $P(\mathbf{T_2})$ **are true for some rooted binary trees $\mathbf{T_1}$ and $\mathbf{T_2}$.**
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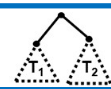
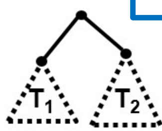
Goal: Prove P().

By defn, size(

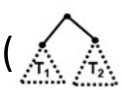
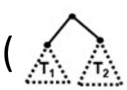
) $=1+\text{size}(\mathbf{T_1})+\text{size}(\mathbf{T_2})$

$\le 1+2^{\text{height}(\mathbf{T_1})+1}-1+2^{\text{height}(\mathbf{T_2})+1}-1$

by IH for $\mathbf{T_1}$ and $\mathbf{T_2}$

$\le 2^{\text{height}(\mathbf{T_1})+1}+2^{\text{height}(\mathbf{T_2})+1}-1$

$\le 2(2^{\max(\text{height}(\mathbf{T_1}),\text{height}(\mathbf{T_2}))+1})-1$

$\le 2(2^{\text{height}(}$  $^{)}) - 1 \le 2^{\text{height}(}$  $^{)+1} - 1$

which is what we wanted to show.

5. **So, the** $P(T)$ **is true for all rooted bin. trees by structural induction.**