CSE 311: Foundations of Computing

Lecture 17: Recursively Defined Sets & Structural Induction
Recursive Definition of Sets

Recursive definition of set $S$

- **Basis Step:** $0 \in S$
- **Recursive Step:** If $x \in S$, then $x + 2 \in S$
- **Exclusion Rule:** Every element in $S$ follows from the basis step and a finite number of recursive steps.

We need the exclusion rule because otherwise $S = \mathbb{N}$ would satisfy the other two parts. However, we won’t always write it down on these slides.
Recursive Definitions of Sets

Basis: \( 6 \in S, \ 15 \in S \)

Recursive: \( \text{If } x, y \in S, \text{ then } x + y \in S \)

Basis: \( [1, 1, 0] \in S, [0, 1, 1] \in S \)

Recursive: \( \text{If } [x, y, z] \in S, \text{ then } [\alpha x, \alpha y, \alpha z] \in S \text{ for any } \alpha \in \mathbb{R} \)

If \( [x_1, y_1, z_1] \in S \) and \( [x_2, y_2, z_2] \in S \), then
\( [x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S \).

Number of form \( 3^n \) for \( n \geq 0 \):

- \( (a, a, 0) \quad a \in \mathbb{R} \)
- \( (0, b, b) \quad b \in \mathbb{R} \)
- \( (a, a + b, b) \quad a, b \in \mathbb{R} \).
Recursive Definitions of Sets

Basis: $6 \in S, 15 \in S$
Recursive: If $x, y \in S$, then $x + y \in S$

Basis: $[1, 1, 0] \in S, [0, 1, 1] \in S$
Recursive: If $[x, y, z] \in S$, then $[\alpha x, \alpha y, \alpha z] \in S$ for any $\alpha \in \mathbb{R}$
If $[x_1, y_1, z_1] \in S$ and $[x_2, y_2, z_2] \in S$, then
$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

Number of form $3^n$ for $n \geq 0$:
Basis: $1 \in S$
Recursive: If $x \in S$, then $3x \in S$. 
Recursive Definitions of Sets: General Form

Recursive definition

- **Basis step**: Some specific elements are in $S$
- **Recursive step**: Given some existing named elements in $S$ some new objects constructed from these named elements are also in $S$.
- **Exclusion rule**: Every element in $S$ follows from the basis step and a finite number of recursive steps
Strings

• **An alphabet** \( \Sigma \) is any finite set of characters

• The set \( \Sigma^* \) of **strings** over the alphabet \( \Sigma \) is defined by
  - **Basis:** \( \varepsilon \in \Sigma^* \) (\( \varepsilon \) is the empty string w/ no chars)
  - **Recursive:** if \( w \in \Sigma^* \), \( a \in \Sigma \), then \( wa \in \Sigma^* \)
Palindromes

Palindromes are strings that are the same backwards and forwards

**Basis:**

ε is a palindrome and any $a \in \Sigma$ is a palindrome

**Recursive step:**

If $p$ is a palindrome then $apa$ is a palindrome for every $a \in \Sigma$
All Binary Strings with no 1’s before 0’s

Define: \( \varepsilon \in S \).

Remove: If \( x \in S \),

- \( 0 \in S \)
- and \( \varepsilon \in S \).

\[ \varepsilon, 0, \ldots, 0^n, 0^n \cdot 1, \ldots, 0^n \cdot 1^n \]

\[ 0^n \cdot 1, \ldots, 0^n \cdot 20. \]
All Binary Strings with no 1’s before 0’s

Basis:
\[ \varepsilon \in S \]

Recursive:
\[ \text{If } x \in S, \text{ then } 0x \in S \]
\[ \text{If } x \in S, \text{ then } x1 \in S \]
Functions on Recursively Defined Sets (on $\Sigma^*$)

Length:
- $\text{len}(\varepsilon) = 0$
- $\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:
- $\varepsilon^R = \varepsilon$
- $(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:
- $x \cdot \varepsilon = x$ for $x \in \Sigma^*$
- $x \cdot wa = (x \cdot w)a$ for $x \in \Sigma^*$, $w \in \Sigma^*$, $a \in \Sigma$

Number of $c$'s in a string:
- $\#_c(\varepsilon) = 0$
- $\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*, a \in \Sigma, a \neq c$
Rooted Binary Trees

• **Basis:**
  - is a rooted binary tree

• **Recursive step:**

If $T_1$ and $T_2$ are rooted binary trees,
then also is a rooted binary tree.
Defining Functions on Rooted Binary Trees

- \( \text{size}(\bullet) = 1 \)

- \( \text{size}\left(\begin{array}{c} T_1 \\ T_2 \end{array}\right) = 1 + \text{size}(T_1) + \text{size}(T_2) \)

- \( \text{height}(\bullet) = 0 \)

- \( \text{height}\left(\begin{array}{c} T_1 \\ T_2 \end{array}\right) = 1 + \max\{\text{height}(T_1), \text{height}(T_2)\} \)
Structural Induction

How to prove $\forall x \in S, P(x)$ is true:

**Base Case:** Show that $P(u)$ is true for all specific elements $u$ of $S$ mentioned in the *Basis step*

**Inductive Hypothesis:** Assume that $P$ is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

**Inductive Step:** Prove that $P(w)$ holds for each of the new elements $w$ constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

**Conclude** that $\forall x \in S, P(x)$
How to prove $\forall x \in S, P(x)$ is true:

**Base Case:** Show that $P(u)$ is true for all specific elements $u$ of $S$ mentioned in the **Basis step**

**Inductive Hypothesis:** Assume that $P$ is true for some arbitrary values of each of the existing named elements mentioned in the **Recursive step**

**Inductive Step:** Prove that $P(w)$ holds for each of the new elements $w$ constructed in the **Recursive step** using the named elements mentioned in the Inductive Hypothesis

**Conclude** that $\forall x \in S, P(x)$
Structural Induction vs. Ordinary Induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

- **Basis:** $0 \in \mathbb{N}$
- **Recursive step:** If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Define $Q(n)$ to be “for all $x \in S$ that can be constructed in at most $n$ recursive steps, $P(x)$ is true.”
Using Structural Induction

Let $S$ be given by...

- **Basis:** $6 \in S; \ 15 \in S;$
- **Recursive:** if $x, y \in S$ then $x + y \in S$.

**Claim:** Every element of $S$ is divisible by 3.
Claim: Every element of $S$ is divisible by 3.

1. Let $P(x)$ be “$3 \mid x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

2. Base Case: $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true.

Basis: $6 \in S;\ 15 \in S;

Recursive: if $x, y \in S$ then $x + y \in S$
Claim: Every element of $S$ is divisible by 3.

1. Let $P(x)$ be “$3 \mid x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

2. Base Case: $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true.

3. Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x, y \in S$.

4. Inductive Step: **Goal: Show** $P(x+y)$

   $3 \mid x$ means $x = 3m$ for some $m$.
   $3 \mid y$ means $y = 3n$ for some $n$.

   \[ x + y = 3m + 3n = 3(m+n), \text{ which says } P(x+y). \]

**Basis:** $6 \in S; 15 \in S$; 

**Recursive:** if $x, y \in S$ then $x + y \in S$. 

Claim: Every element of $S$ is divisible by 3.

1. Let $P(x)$ be “$3 \mid x$”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.

2. Base Case: $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true.

3. Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x, y \in S$.

4. Inductive Step: **Goal: Show $P(x+y)$**

   Since $P(x)$ is true, $3 \mid x$ and so $x=3m$ for some integer $m$ and since $P(y)$ is true, $3 \mid y$ and so $y=3n$ for some integer $n$.

   Therefore $x+y=3m+3n=3(m+n)$ and thus $3 \mid (x+y)$.

   Hence $P(x+y)$ is true.

5. Therefore by induction $3 \mid x$ for all $x \in S$.

**Basis:** $6 \in S; \ 15 \in S$;

**Recursive:** if $x, y \in S$ then $x + y \in S$
Claim: \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

Let \( P(y) \) be “\( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x \in \Sigma^* \)”.
We prove \( P(y) \) for all \( y \in \Sigma^* \) by structural induction.

**Base Case:** \( \text{len}(x \cdot \varepsilon) = \text{len}(x) \) by def of \( \cdot \)

\[
= \text{len}(x) + 0 \\
= \text{len}(x) + \text{len}(\varepsilon) \quad \text{defn of len.}
\]

This proves \( P(\varepsilon) \).

\( \varepsilon \in \Sigma^+ \)

\( w \varepsilon \in \Sigma^0 \)

\[
\text{len}(\varepsilon) = 0 \\
\text{len}(w \varepsilon) = \text{len}(w) + 1 \\
x \cdot \varepsilon = x \\
x \cdot (w \varepsilon) = (x \cdot w) \varepsilon.
\]
Claim: \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

Let \( P(y) \) be “\( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x \in \Sigma^* \)”.

We prove \( P(y) \) for all \( y \in \Sigma^* \) by structural induction.

Base Case: \( y = \varepsilon \). For any \( x \in \Sigma^* \), \( \text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon) \) since \( \text{len}(\varepsilon) = 0 \). Therefore \( P(\varepsilon) \) is true.
Claim: \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

Let \( P(y) \) be “\( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x \in \Sigma^* \)”.
We prove \( P(y) \) for all \( y \in \Sigma^* \) by structural induction.

Base Case: \( y = \varepsilon \). For any \( x \in \Sigma^* \), \( \text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon) \) since \( \text{len}(\varepsilon) = 0 \). Therefore \( P(\varepsilon) \) is true.

Inductive Hypothesis: Assume that \( P(w) \) is true for some arbitrary \( w \in \Sigma^* \).

Inductive Step: Goal: Show that \( P(wa) \) is true for every \( a \in \Sigma \).

Let \( w \in \Sigma^* \) be arbitrary.

\[
\begin{align*}
\text{len}(x \cdot (wa)) &= \text{len}((x \cdot w) \cdot a) \\
&= \text{len}(x \cdot w) + 1 \\
&= \text{len}(x) + \text{len}(w) + 1 \\
&= \text{len}(x) + \text{len}(wa)
\end{align*}
\]

Therefore, \( P(wa) \) is true by induction.

Claim: \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)
Claim: $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x, y \in \Sigma^*$

Let $P(y)$ be “$\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$”. We prove $P(y)$ for all $y \in \Sigma^*$ by structural induction.

Base Case: $y = \varepsilon$. For any $x \in \Sigma^*$, $\text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ since $\text{len}(\varepsilon) = 0$. Therefore $P(\varepsilon)$ is true.

Inductive Hypothesis: Assume that $P(w)$ is true for some arbitrary $w \in \Sigma^*$.

Inductive Step: **Goal: Show that $P(wa)$ is true for every $a \in \Sigma$**

Let $a \in \Sigma$. Let $x \in \Sigma^*$. Then $\text{len}(x \cdot wa) = \text{len}((x \cdot w)a)$ by defn of $\cdot$

$= \text{len}(x \cdot w) + 1$ by defn of $\text{len}$

$= \text{len}(x) + \text{len}(w) + 1$ by I.H.

$= \text{len}(x) + \text{len}(wa)$ by defn of $\text{len}$

Therefore $\text{len}(x \cdot wa) = \text{len}(x) + \text{len}(wa)$ for all $x \in \Sigma^*$, so $P(wa)$ is true.

So, by induction $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x, y \in \Sigma^*$.
Claim: For every rooted binary tree $T$, $\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$
Claim: For every rooted binary tree \( T \), \( \text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1 \)

1. Let \( P(T) \) be “\( \text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1 \)”.
   We prove \( P(T) \) for all rooted binary trees \( T \) by structural induction.

2. Base Case: \( \text{size}(\cdot) = 1 \), \( \text{height}(\cdot) = 0 \) and
   \( 1 = 2^1 - 1 = 2^{0+1} - 1 \) so \( P(\cdot) \) is true.
Claim: For every rooted binary tree $T$, $\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$

1. Let $P(T)$ be “$\text{size}(T) \leq 2^{\text{height}(T)}+1-1$”. We prove $P(T)$ for all rooted binary trees $T$ by structural induction.

2. Base Case: $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ and $1=2^1-1=2^{0+1}-1$ so $P(\bullet)$ is true.

3. Inductive Hypothesis: Suppose that $P(T_1)$ and $P(T_2)$ are true for some rooted binary trees $T_1$ and $T_2$.

4. Inductive Step: Goal: Prove $P(\begin{array}{c} T_1 \\ / \\ / \\ T_2 \end{array})$. 

So, the $P(T)$ is true for all rooted bin. trees by structural induction.
**Claim:** For every rooted binary tree $T$, $\text{size}(T) \leq 2^{\text{height}(T)} + 1 - 1$

1. Let $P(T)$ be “$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$”. We prove $P(T)$ for all rooted binary trees $T$ by structural induction.

2. Base Case: $\text{size}(\bullet) = 1$, $\text{height}(\bullet) = 0$ and $1 = 2^1 - 1 = 2^{0+1} - 1$ so $P(\bullet)$ is true.

3. Inductive Hypothesis: Suppose that $P(T_1)$ and $P(T_2)$ are true for some rooted binary trees $T_1$ and $T_2$.

4. Inductive Step:  

   **Goal:** Prove $P(\begin{array}{c}
   \text{\textbullet} \\
   \text{T}_1 \\
   \text{T}_2
   \end{array})$.

   By defn, $\text{size}(\begin{array}{c}
   \text{\textbullet} \\
   \text{T}_1 \\
   \text{T}_2
   \end{array}) = 1 + \text{size}(T_1) + \text{size}(T_2)$

   $\leq 1 + 2^{\text{height}(T_1)+1} - 1 + 2^{\text{height}(T_2)+1} - 1$

   by IH for $T_1$ and $T_2$

   $\leq 2^{\text{height}(T_1)+1} + 2^{\text{height}(T_2)+1} - 1$

   $\leq 2\left(\max(\text{height}(T_1),\text{height}(T_2))+1\right) - 1$

   $\leq 2\left(2^{\text{height}}\right)-1 \leq 2^{\text{height}+1} - 1$

   which is what we wanted to show.

5. So, the $P(T)$ is true for all rooted bin. trees by structural induction.