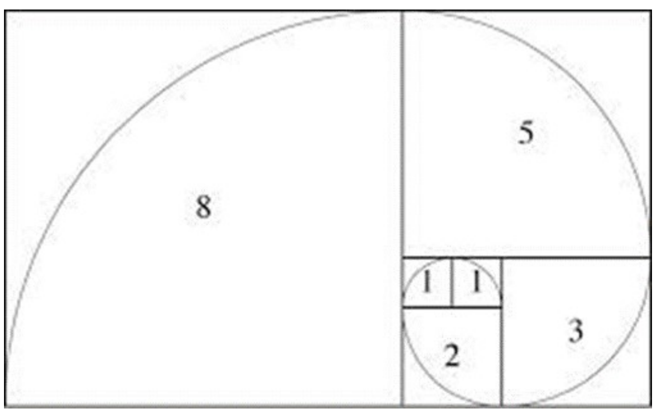
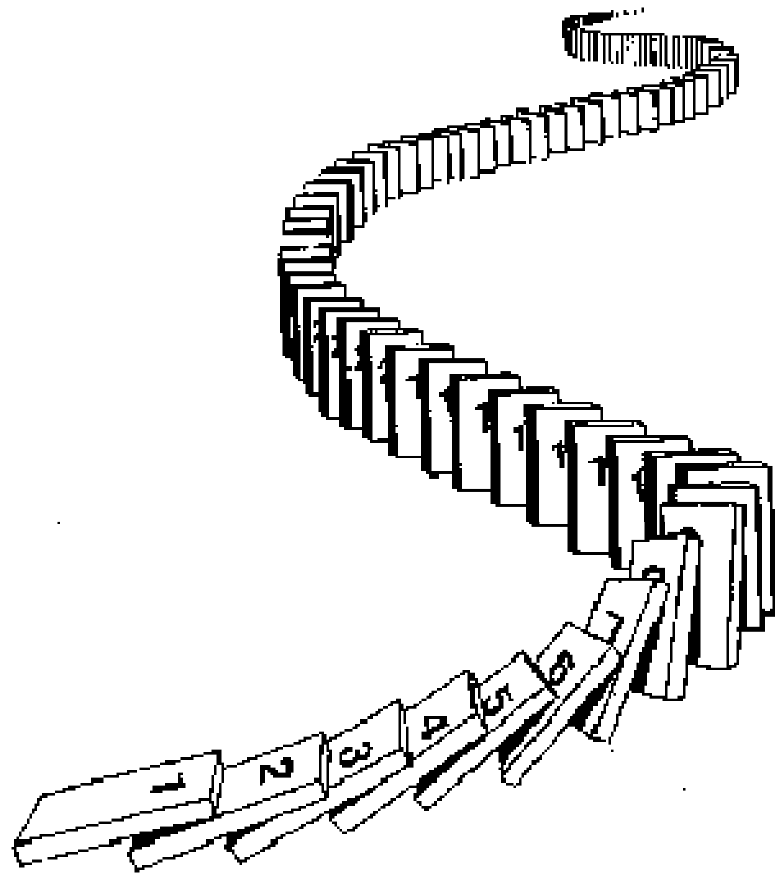
CSE 311: Foundations of Computing

Lecture 16: Recursion & Strong Induction Applications: Fibonacci & Euclid





Midterm

- A week today (Monday, May 7) in class
- Closed book, closed notes
  - You will get lists of inference rules & equivalences
- Covers material up to end of ordinary induction.
- Practice problems & practice midterm on the website
  - Solutions later this week

- Prof. Beame will run a review session
  Sunday, May 6, 3:30-5:30 pm in EEB 105.

More Recursive Definitions

Suppose that $h: \mathbb{N} \to \mathbb{R}$.

$S(n)$

$S(0) = h(0)$

$S(n+1) = h(n+1) + S(n)$

Then we have familiar summation notation:

$\sum_{i=0}^{0} h(i) = h(0)$

$\sum_{i=0}^{n+1} h(i) = h(n+1) + \sum_{i=0}^{n} h(i)$ for $n \geq 0$

$\sum_{i=0}^{n} h(i) = h(0) + h(1) \ldots + h(n)$

There is also product notation:

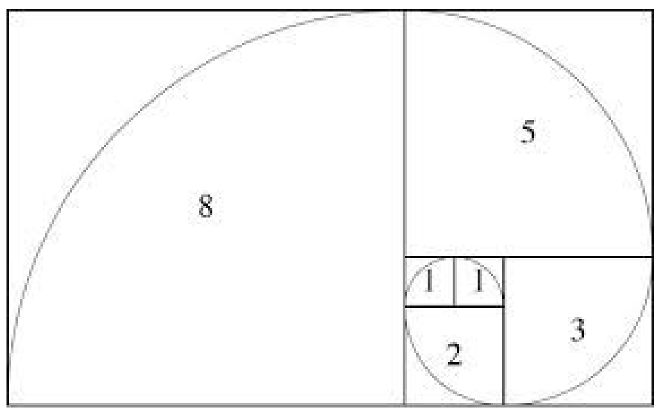
$\prod_{i=0}^{0} h(i) = h(0)$

$\prod_{i=0}^{n+1} h(i) = h(n+1) \cdot \prod_{i=0}^{n} h(i)$ for $n \geq 0$

Fibonacci Numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$\underbrace{f_n = f_{n-1} + f_{n-2}}$$ **for all** $n \geq 2$

$n - 2 \geq 0.$



Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ is true for every integer j from b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let $P(n)$ be "$f_n < 2^n$". We will prove $P(n)$ for all $n \geq 0$ by <u>strong</u> induction.

2. Base Case ($n=0$). $f_0 = 0 < 1 = 2^0$ so $P(0)$ is true.

$f_0 = 0$ $\quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0=0 < 1= 2^0$ so P(0) is true.

3. IH: Suppose P(j) is true for all $j = 0..k$, for an arbitrary k.

4. Induction Step: Show P(k+1) is true.

i.e. $f_{k+1} < 2^{k+1}$

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0 = 0 < 1 = 2^0$ so P(0) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, P(j) is true for every integer j from 0 to k.
4. Inductive Step: $\boxed{\text{Goal: Show } P(k+1)\text{; that is, } f_{k+1} < 2^{k+1}}$

If $k+1 = 1$; $f_{k+1} = f_1 = 1 < 2^1 = 2^{k+1}$.

otherwise $k+1 > 1$. $f_{k+1} = f_k + f_{k-1}$

$< 2^k + 2^{k-1}$ by IH.

$< 2 \cdot 2^k = 2^{k+1}$

$f_{0,1} = f_{-1}$

$f_{k+1} = f_k + f_{k-1}$

In either case, $f_{k+1} < 2^{k+1}$ so P(k+1) holds.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0=0 < 1= 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$, P(j) **is true for every integer** j **from** 0 **to** k**.**
4. **Inductive Step:** **Goal: Show** P(k+1)**; that is,** $f_{k+1} < 2^{k+1}$

   <u>Case k+1 = 1</u>:

   <u>Case k+1 ≥ 2</u>:

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** $n \geq 0$ **by strong induction.**
2. **Base Case:** $f_0 = 0 < 1 = 2^0$ **so** P(0) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 0$, P(j) **is true for every integer** j **from** 0 **to** k.
4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} < 2^{k+1}}$

   <u>Case k+1 = 1</u>**: Then** $f_1 = 1 < 2 = 2^1$ **so** P(k+1) **is true here.**

   <u>Case k+1 ≥ 2</u>**: Then** $f_{k+1} = f_k + f_{k-1}$ **by definition**

   $< 2^k + 2^{k-1}$ **by the IH since** $k-1 \geq 0$

   $< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

   so P(k+1) **is true in this case.**

   These are the only cases so P(k+1) **follows.**
5. **Therefore by strong induction,**

   $f_n < 2^n$ **for all integers** $n \geq 0$.

$$\boxed{\begin{array}{l} f_0 = 0 \qquad f_1 = 1 \\ f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2 \end{array}}$$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. Let $P(n)$ be "$f_n \geq 2^{n/2-1}$". We will prove $P(n)$ for all n by strong induction.

2. Base case ($n=2$): $f_2 = f_1 + f_0 = 1 + 0 = 1$ and $2^{2/2-1} = 2^{1-1} = 2^0 = 1$. These are equal, so $P(2)$ is true.

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2 - 1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2 - 1} = 2^0 = 1$ **so** P(2) **is true.**

3. IH: for an arbitrary $k \geq 2$, suppose that $P(j)$ holds for $j = 2..k$.

4. Inductive Step. Show $P(k+1)$, i.e.

$$f_{k+1} \geq 2^{(k+1)/2 - 1}$$

if $k+1 = 3$

$k - 1 = 1$.

Want to $f_{k+1} = f_k + f_{k-1}$

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so P(2) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, P(j) is true for every integer j from 2 to k.
4. Inductive Step: Goal: Show P(k+1); that is, $f_{k+1} \geq 2^{(k+1)/2-1}$

No need for cases for the definition here:

$f_{k+1} = f_k + f_{k-1}$ since $k+1 \geq 2$

Now just want to apply the IH to get P(k) and P(k-1):

Problem: Though we can get P(k) since $k \geq 2$,

k-1 may only be 1 so we can't conclude P(k-1)

Solution: Separate cases for when k-1=1 (or k+1=3).

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2-1}$**". We prove that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** $k \geq 2$, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; that is, } f_{k+1} \geq 2^{(k+1)/2-1}}$

$k+1=3$ <u>Case k = 2</u>: $f_{k+1} = f_3 = f_2 + f_1 = 1 + 1 = 2$

Also, $2^{(k+1)/2-1} = 2^{3/2-1} = 2^{1/2}$

Since $2 \geq 2^{1/2}$, P(3) holds

$k+1 \geq 4$ <u>Case k ≥ 3</u>:

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2 - 1}$ for all $n \geq 2$

1. **Let** P(n) **be "**$f_n \geq 2^{n/2 - 1}$**". We prove that** P(n) **is true for all integers** n ≥ 2 **by strong induction.**
2. **Base Case:** $f_2 = f_1 + f_0 = 1$ and $2^{2/2 - 1} = 2^0 = 1$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Assume that for some arbitrary integer** k ≥ 2, P(j) **is true for every integer** j **from** 2 **to** k.
4. **Inductive Step:** **Goal: Show** P(k+1)**; that is,** $f_{k+1} \geq 2^{(k+1)/2 - 1}$

   Case k = 2: **Then** $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2 - 1}$

   Case k ≥ 3:

$$f_{k+1} = f_k + f_{k-1} \quad \text{(note: IH applies)}$$
$$\geq 2^{k/2 - 1} + 2^{(k-1)/2 - 1} \quad \text{by IH.}$$
$$\geq 2 \cdot 2^{(k-1)/2 - 1} = 2^{(k-1)/2} = 2^{(k+1)/2 - 1}$$

$$(k+1)/2 - 1$$
$$= (k+1-2)/2$$
$$= (k-1)/2.$$

In either case, we see that P(k+1) is true.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let $P(n)$ be "$f_n \geq 2^{n/2-1}$". We prove that $P(n)$ is true for all integers $n \geq 2$ by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1$ and $2^{2/2-1} = 2^0 = 1$ so $P(2)$ is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, $P(j)$ is true for every integer j from 2 to k.
4. Inductive Step: Goal: Show $P(k+1)$; that is, $f_{k+1} \geq 2^{(k+1)/2-1}$

   Case $k = 2$: Then $f_{k+1} = f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1} = 2^{(k+1)/2-1}$

   Case $k \geq 3$: $f_{k+1} = f_k + f_{k-1}$ by definition

   $\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ by the IH since $k-1 \geq 2$

   $\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2^{(k-1)/2} = 2^{(k+1)/2-1}$

   So $P(k+1)$ is true in both cases.
5. Therefore by strong induction, $f_n \geq 2^{n/2-1}$ for all integers $n \geq 0$.

$f_0 = 0 \quad f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes** n **steps for** $\gcd(a, b)$ **with** $a \geq b > 0$. **Then,** $a \geq f_{n+1}$.

An informal way to get the idea: Consider an n **step** gcd **calculation starting with** r_{n+1}=a **and** r_n=b:

$a = qb + r$ (a,b)

$b = q'r + r'$ (b,r)

(r,r')

$$r_{n+1} = q_n r_n + r_{n-1}$$
$$r_n = q_{n-1} r_{n-1} + r_{n-2}$$
$$\ldots$$
$$r_3 = q_2 r_2 + r_1$$
$$r_2 = q_1 r_1$$

$a = r_{n+1}$

$b = r_n$

For all k ≥ 2, $r_{k-1} = r_{k+1} \bmod r_k$

fib.

$r_{n+1} \geq r_n + r_{n-1}$

$r_n \geq r_{n-1} + r_{n-2}$

Now $r_1 \geq 1$ **and each** q_k **must be** ≥ 1. **If we replace all the** q_K's **by** 1 **and replace** r_1 **by** 1 **, we can only reduce the** r_k's. **After that reduction,** $r_k = f_k$ **for every** k.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

We go by strong induction on n**.**

1. **Let** P(n) **be "**gcd(a,b) **with** a ≥ b>0 **takes** n **steps** → a ≥ f_{n+1}**" for all** n ≥ 1**.**

2. **<u>Base Case</u>:** n=1 **If Euclid's Algorithm on** a, b **with** a ≥ b > 0 **takes** 1 **step, then** a=q_1b **for some** q_1 **and** a ≥ b ≥ 1=f_2 **and** P(1) **holds**

<u>Induction Hypothesis</u>: Suppose that for some integer k ≥ 1, P(j) **is true for all integers** j **s.t.** 1 ≤ j ≤ k

<u>Inductive Step</u>: We want to show: if gcd(a,b) **with** a ≥ b > 0 **takes** k+1 **steps, then** a ≥ $f_{k+2.}$

$f_{k+1} = f_k + f_{k-1}$

Running time of Euclid's algorithm

<u>Induction Hypothesis</u>: Suppose that for some integer $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>Inductive Step</u>: We want to show: if $\gcd(a,b)$ **with** $a \geq b>0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2}$.

Now if $k=1$, **the two steps of Euclid's algorithm on** a **and** b **are given by** $\gcd(a,b)=\gcd(b,c)=\gcd(c,0)=c$ **where**

$a = q_2 b + c$ $\quad c > 0$ $\quad a \geq f_3 = f_{k+2}$

$b = q_1 c$

and $c = a \bmod b > 0$

Also, since $a \geq b$ **we must have** $q_2 \geq 1$.

So $a = q_2 b + c \geq b + c \geq 1+1 = 2 = f_3 = f_{k+2}$ as required.

$q_2 \geq 1.$

Running time of Euclid's algorithm

takes j steps → $a \geq f_{j+1}$

Induction Hypothesis: Suppose that for some integer $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

Inductive Step: We want to show: if gcd(a,b) **with** $a \geq b>0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2.}$

Next suppose that $k \geq 2$ **so for the first three steps of Euclid's algorithm on** a **and** b **we have** gcd(a,b)=gcd(b,c)=gcd(c,d) **where**

$a = q_{k+1}b + c$ ← $c>0$ step $k+1$

$b = q_k c + d$ ← $d>0$ step k — $b \geq f_{k+1}$ by IH

$c = q_{k-1}d + e$ (c = a mod b , d = b mod c , e = c mod d and d>0)

step $k-1$ $c \geq f_k$

By definition of mod we have $b > c > d>0$, gcd(b,c) **takes** k **steps and** gcd(c,d) **takes** $k-1 \geq 1$ **steps, so by the IH we have** $b \geq f_{k+1}$ **and** $c \geq f_k$.

Also, since $a \geq b$ **we must have** $q_{k+1} \geq 1$.

So $a = q_{k+1}b + c \geq b + c \geq f_{k+1}+ f_k= f_{k+2}$ as required.

IH

$a \geq f_{k+2}$

$f_{(k+1)+1}$

$P(k+1)$ is true.

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Why does this help us bound the running time of Euclid's Algorithm?

We already proved that $f_n \geq 2^{n/2 - 1}$ so $f_{n+1} \geq 2^{(n-1)/2}$

Therefore: if Euclid's Algorithm takes n steps
for $\gcd(a, b)$ with $a \geq b > 0$
then $a \geq 2^{(n-1)/2}$

$O(\log_2 a)$

so $(n-1)/2 \leq \log_2 a$ or $\boxed{n \leq 1 + 2\log_2 a}$
i.e., # of steps $\leq$ twice the # of bits in a.

Recursive Definition of Sets

Recursive Definition

- Basis Step: $0 \in S$
- Recursive Step: If $x \in S$, then $x + 2 \in S$
- Exclusion Rule: Every element in S follows from basis steps and a finite number of recursive steps.

$0, 2, 4, 6, \ldots$

Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$

Recursive: If $x,y \in S$, then $x+y \in S$

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$

Recursive: If $[x, y, z] \in S$, then $[\alpha x, \alpha y, \alpha z] \in S$ for any $\alpha \in \mathbb{R}$

If $[x_1, y_1, z_1] \in S$ and $[x_2, y_2, z_2] \in S$, then

$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

$\{a, a+b, 5\}$

Powers of 3:

Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$

Recursive: If $x,y \in S$, then $x+y \in S$

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$

Recursive: If $[x, y, z] \in S$, then $[\alpha x, \alpha y, \alpha z] \in S$

If $[x_1, y_1, z_1] \in S$ and $[x_2, y_2, z_2] \in S$, then

$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

Powers of 3:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$.

Recursive Definitions of Sets: General Form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma$ (ε is the empty string)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same backwards and forwards

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$

All Binary Strings with no 1's before 0's

All Binary Strings with no 1's before 0's

Basis:

$\mathcal{E} \in S$

Recursive:

If x $\in S$, **then** 0x $\in S$

If x $\in S$, **then** x1 $\in S$

Function Definitions on Recursively Defined Sets

Length:

$\text{len}(\mathcal{E}) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\mathcal{E}^R = \mathcal{E}$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \mathcal{E} = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$