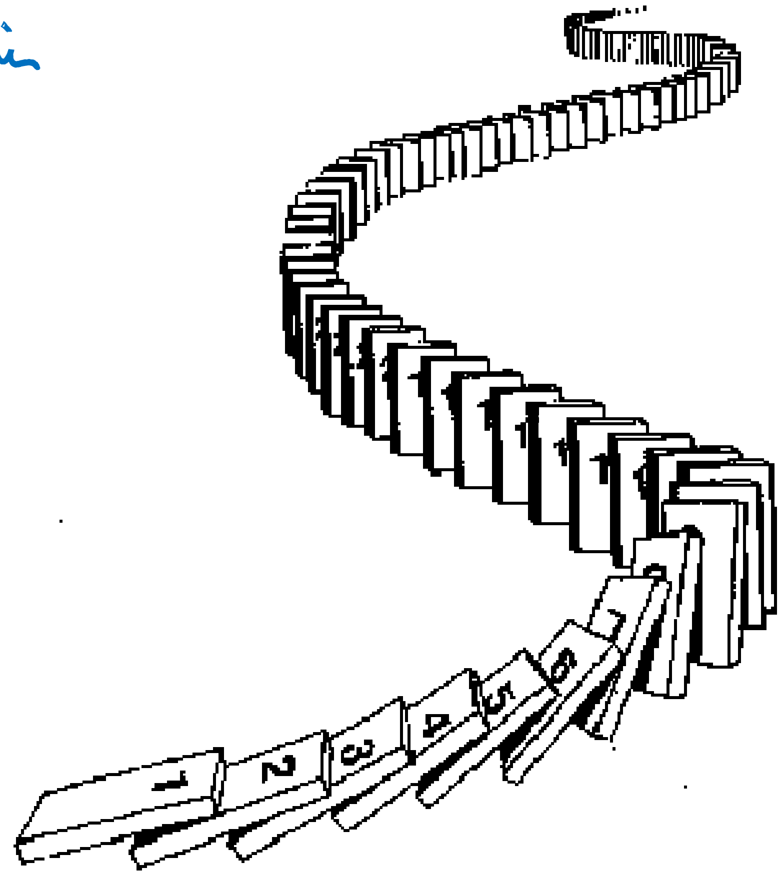
CSE 311: Foundations of Computing

Lecture 15: Induction & Strong Induction

Do one of the practice induction problems from section

& see section solutions



Inductive Proofs In 5 Easy Steps

1. **"Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq 0$ by induction."**
2. **"Base Case:" Prove $P(0)$**
3. **"Inductive Hypothesis:**

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$"
4. **"Inductive Step:" Prove that $P(k+1)$ is true:**

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. **"Conclusion: $P(n)$ is true for all integers $n \geq 0$"**

Induction: Changing the start line

- What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b?

- Define predicate $Q(k) = P(k + b)$ for all k.
  - Then $\forall n \geq 0\ Q(n) \equiv \forall n \geq b\ \ P(n)$

- Ordinary induction for Q:
  - Prove $Q(0) \equiv P(b)$
  - Prove
    $\forall k \geq 0 \big(Q(k) \longrightarrow Q(k+1)\big) \equiv \forall k \geq b\big(P(k) \longrightarrow P(k+1)\big)$

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq \boldsymbol{b}$ by induction."
2. "Base Case:" Prove $P(\boldsymbol{b})$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq \boldsymbol{b}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \ \geq \boldsymbol{b}$ by induction."
2. "Base Case:" Prove $P(\boldsymbol{b})$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \ \geq \boldsymbol{b}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let $P(n)$ be $3^n \geq n^2+3$. We will show that $P(n)$ is ~~true~~ for all integers $n \geq 2$ by induction

2. <u>Base Case</u>: $\boxed{n=2}$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be "**$3^n \geq n^2+3$**". We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 2^2+3$ ✓
3. Inductive Hypothesis: Assume for some arbitrary integer $k \geq 2$ that P(k) is true (i.e. $3^k \geq k^2+3$)
4. Inductive Step:

$$\boxed{\text{Goal: Show } P(k+1) \text{ is } \cancel{\text{true}} \quad 3^{k+1} \geq (k+1)^2+3}$$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$**.**
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3 = k^2+2k+4$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** n ≥ 2 **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 2**.**
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3 = k^2+2k+4$

$$
\begin{aligned}
3^{k+1} &= 3 \cdot 3^k \\
&\geq 3(k^2+3) \quad \text{by IH } (*) \\
&= 3k^2+9 \\
&= k^2+2k^2+9 \\
&\geq k^2+2k+9 \quad \text{since } k \geq 1 \\
&\geq k^2+2k+4
\end{aligned}
$$

$\therefore$ P(k+1) is true

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$.

   $3^k \geq k^2+3$

4. **Inductive Step:**

   Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

   $$
   \begin{aligned}
   3^{k+1} &= 3(3^k) \\
   &\geq 3(k^2+3) \text{ by the IH} \\
   &= k^2+2k^2+9 \\
   &\geq k^2+2k+4 = (k+1)^2+3 \text{ since } k \geq 1.
   \end{aligned}
   $$

   Therefore P(k+1) **is true.**

5. **Thus** P(n) **is true for all integers** $n \geq 2$, **by induction.**

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$P(0) \xrightarrow{P(0)\to P(1)} P(1) \xrightarrow{P(1)\to P(2)} P(2) \xrightarrow{P(2)\to P(3)} P(3) \xrightarrow{P(3)\to P(4)} P(4) \xrightarrow{P(4)\to P(5)} P(5)$$

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$\begin{tikzcd} P(0) \arrow[r, bend left, "P(0)\to P(1)"] & P(1) \arrow[r, bend left, "P(1)\to P(2)"] & P(2) \arrow[r, bend left, "P(2)\to P(3)"] & P(3) \arrow[r, bend left, "P(3)\to P(4)"] & P(4) \arrow[r, bend left, "P(4)\to P(5)"] & P(5) \end{tikzcd}$$

We made it harder than we needed to …

When we proved $P(2)$ we knew BOTH $P(0)$ and $P(1)$

When we proved $P(3)$ we knew $P(0)$ and $P(1)$ and $P(2)$

When we proved $P(4)$ we knew $P(0)$, $P(1)$, $P(2)$, $P(3)$

etc.

That's the essence of the idea of Strong Induction.

Strong Induction

$$P(0)$$

$$\forall k\ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \rightarrow P(k+1)\Big)$$

$$\therefore \forall n\ P(n)$$

Strong Induction

$$P(0)$$
$$\forall k\ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \to P(k+1)\Big)$$

$$\therefore\ \forall n\ P(n)$$

Strong induction for P follows from ordinary induction for Q where

$$Q(k) = P(0) \land P(1) \land P(2) \land \cdots \land P(k)$$

Note that $Q(0) \equiv P(0)$ and $Q(k+1) \equiv Q(k)\ \land P(k+1)$ and $\forall n\ Q(n) \equiv \forall n\ P(n)$

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(k)$ is true"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

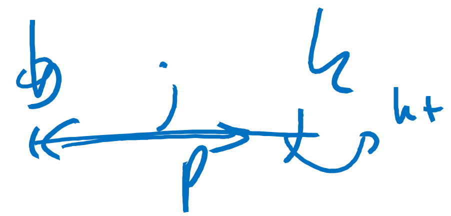
   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)

5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq b$ by ***strong*** induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:



   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ is true for every integer j from b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

48 = 2 • 2 • 2 • 2 • 3
591 = 3 • 197
45,523 = 45,523
321,950 = 2 • 5 • 5 • 47 • 137
1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

We use strong induction to prove that a factorization into primes exists, but not that it is unique.

Every integer ≥ 2 is a product of primes.

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**

3. Inductive Hypothesis: Assume that ~~for~~ some arbitrary integer $k \geq 2$ every integer j with $2 \leq j \leq k$ is a product of primes.

4. Inductive Step: Goal: Show $k+1$ is a product

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n $\geq$ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** k ≥ 2,
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

$$\boxed{\textbf{Goal: Show } P(k+1)\textbf{; i.e. } k+1 \textbf{ is a product of primes}}$$

Case: k+1 is prime: ∴ it is a product of primes k+1 ✓

Case k+1 is composite: ∴ k+1 = a · b for a, b integers 1 < a, b ≤ k+1

∴ 2 ≤ a, b ≤ k

∴ P(a), P(b) true by IH

a = $p_1 \cdots p_r$

b = $\underbrace{q_1 \cdots q_s}_{\text{primes}}$

k+1 = $\underbrace{p_1 \cdots p_r q_1 \cdots q_s}_{\text{primes}}$ ∴ P(k+1)

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) i**s true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$**,**
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

   <u>**Case:** k+1 **is prime**</u>**: Then by definition** k+1 **is a product of primes**

Every integer ≥ 2 is a product of primes.

1. **Let** $P(n)$ **be** "n is a product of primes". **We will show that** $P(n)$ **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore $P(2)$ **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$**,**
   $P(j)$ **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; i.e. } k+1 \textbf{ is a product of primes}}$

   Case: $k+1$ **is prime: Then by definition** $k+1$ **is a product of primes**

   Case: $k+1$ **is composite: Then** $k+1=ab$ **for some integers** a **and** b **where** $2 \leq a, b \leq k$.

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
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   <u>**Case:** k+1 **is prime**</u>**: Then by definition** k+1 **is a product of primes**

   <u>**Case:** k+1 **is composite:**</u> **Then** k+1=ab **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** P(a) **and** P(b) **are true so we have**

   $a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$

   for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

   Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

Every integer ≥ 2 is a product of primes.

1. **Let** $P(n)$ **be** "n is a product of primes". **We will show that** $P(n)$ **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore $P(2)$ **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$, $P(j)$ **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show $P(k+1)$; **i.e.** $k+1$ **is a product of primes**

   <u>**Case:** $k+1$ **is prime**</u>: **Then by definition** $k+1$ **is a product of primes**

   <u>**Case:** $k+1$ **is composite:**</u> **Then** $k+1=ab$ **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** $P(a)$ **and** $P(b)$ **are true so we have**

   $a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$

   for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

   Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

   Since $k \geq 2$, **one of these cases must happen and so** $P(k+1)$ **is true:**
5. **Thus** $P(n)$ **is true for all integers** $n \geq 2$, **by strong induction.**

Strong Induction is particularly useful when…

…we need to analyze methods that on input k make a recursive call for an input different from $k-1$.

e.g.: Recursive Modular Exponentiation:

- For exponent $k > 0$ it made a recursive call with exponent $\mathrm{j} = k/2$ when k was even or $\mathrm{j} = k-1$ when k was odd.

We won't analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.

Recursive definitions of functions

- $F(0) = 0;\;\; F(n+1) = F(n) + 1$ **for all** $n \geq\; 0$.

  $F(n) = n$ for all $n \geq 0$

- $G(0) = 1;\;\; G(n+1) = 2 \cdot G(n)$ **for all** $n \geq\; 0$.

  $G(n) = 2^n$ $\forall$ all $n \geq 0$

- $0! = 1;\;\; (n+1)! = (n+1) \cdot n!$ **for all** $n \geq\; 0$.

  $1! = 1$ $3! = 6$ $4! = 24$ $5! = 120$.
  $2! = 2$

- $H(0) = 1;\;\; H(n+1) = 2^{H(n)}$ **for all** $n \geq\; 0$.

  $H(n) = 2^{2^{2^{2^{2}}}}$ n

Prove $n! \leq n^n$ for all $n \geq 1$

$0! = 1$

$(n+1)! = (n+1)\,n!$

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "$n! \leq n^n$". **We will show that** P(n) **is true for all integers** $n \geq 1$ **by induction.**
2. **Base Case** (n=1): $1! = 1 \cdot 0! = 1 \cdot 1 = 1 = 1^1$ **so** P(1) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 1$**.** i.e. $k! \leq k^k$
4. **Inductive Step:**

$\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } (k+1)! \leq (k+1)^{k+1}}$

$$
\begin{aligned}
(k+1)! &= (k+1) \cdot k! && \textbf{by definition of !} \\
&\leq (k+1) \cdot k^k && \textbf{by the IH} \\
&\leq (k+1) \cdot (k+1)^k && \textbf{since } k \geq 0 \\
&= (k+1)^{k+1}
\end{aligned}
$$

Therefore P(k+1) **is true.**

5. **Thus** P(n) **is true for all** $n \geq 1$**, by induction.**