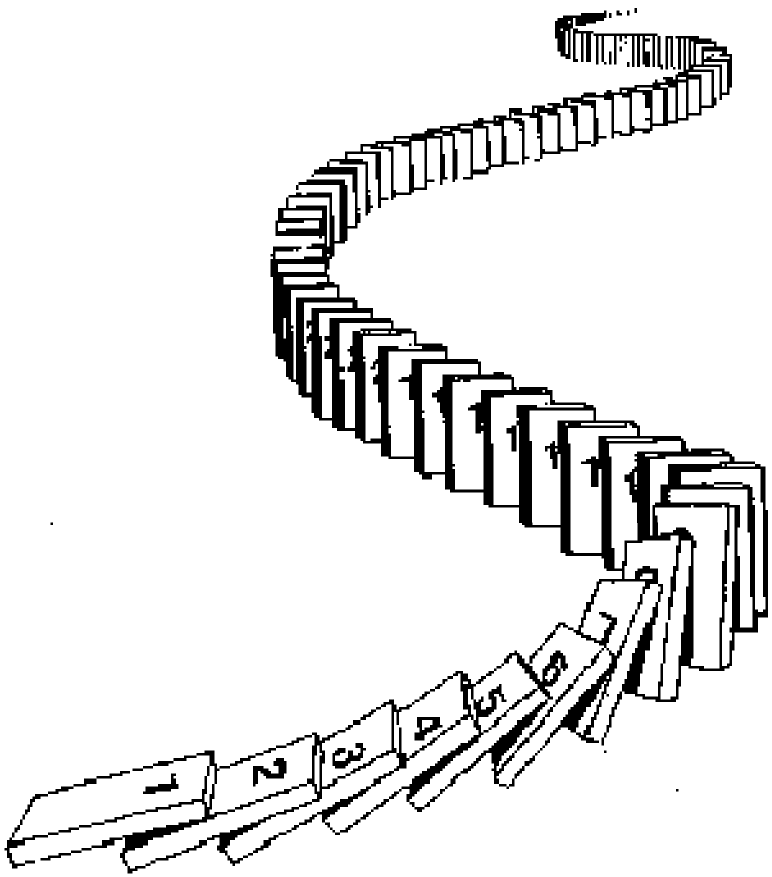
CSE 311: Foundations of Computing

Lecture 15: Induction & Strong Induction



Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq 0$ by induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq 0$"

$\forall k \; P(k) \rightarrow P(k+1)$

Induction: Changing the start line

- What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b?

- Define predicate $Q(k) = P(k + b)$ for all k.
  - Then $\forall n\, Q(n) \equiv \forall n \geq b\;\; P(n)$

$Q(0) = P(b)$

- Ordinary induction for Q:
  - Prove $Q(0) \equiv P(b)$
  - Prove

$$\forall k\, \big(Q(k) \longrightarrow Q(k+1)\big) \equiv \forall k \geq b \big(P(k) \longrightarrow P(k+1)\big)$$

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq \boldsymbol{b}$ by induction."
2. "Base Case:" Prove $P(\boldsymbol{b})$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $k \geq \boldsymbol{b}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Suppose that, for all $i \in \mathbb{N}$, we have $a_i \leq b_i$.
Prove that $a_1 \cdots a_n \leq b_1 \cdots b_n$ for all $n \geq 1$.

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1. **Let** P(n) **be** "$a_1 \cdots a_n \leq b_1 \cdots b_n$". **We will show** P(n) **is true for all integers** n ≥ 1 **by induction.**

Base Case (n=1): $a_1 \leq b_1$ given ($\forall$ Elim)

Suppose that, for all $i \in \mathbb{N}$, we have $a_i \leq b_i$.
Prove that $a_1 \cdots a_n \leq b_1 \cdots b_n$ for all $n \geq 1$.

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1. **Let** P(n) **be** "$a_1 \cdots a_n \leq b_1 \cdots b_n$". **We will show** P(n) **is true for all integers** n ≥ 1 **by induction.**
2. Base Case (n=1): $a_1 \leq b_1$ **is given, so** P(1) **is true.**

Suppose that, for all $i \in \mathbb{N}$, we have $a_i \leq b_i$. ← $\forall i \geq 1$ $a_i \leq b_i$ ($0 \leq$)
Prove that $a_1 \cdots a_n \leq b_1 \cdots b_n$ for all $n \geq 1$.

---

1. **Let** P(n) **be** "$a_1 \cdots a_n \leq b_1 \cdots b_n$". **We will show** P(n) **is true for all integers** n ≥ 1 **by induction.**
2. Base Case (n=1): $a_1 \leq b_1$ **is given, so** P(1) **is true.**
3. Inductive Hypothesis: for an arbitrary integer k ≥ 1,
   suppose that P(k) is true (i.e., "$a_1 \cdots a_k \leq b_1 \cdots b_k$").
4. Inductive Step:

   Goal: show P(k+1), i.e., "$a_1 \cdots a_{k+1} \leq b_1 \cdots b_{k+1}$"

First, note that $a_{k+1} \leq b_{k+1}$. (given)

$a_1 \cdots a_{k+1} = (a_1 \cdots a_k) a_{k+1}$
$\leq (b_1 \cdots b_k) a_{k+1}$ IH
$\leq b_1 \cdots b_k b_{k+1}$ by above.

Suppose that, for all $i \in \mathbb{N}$, we have $a_i \leq b_i$.
Prove that $a_1 \cdots a_n \leq b_1 \cdots b_n$ for all $n \geq 1$.

---

1. **Let** P(n) **be** "$a_1 \cdots a_n \leq b_1 \cdots b_n$". **We will show** P(n) **is true for all integers** n ≥ 1 **by induction.**
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   suppose that P(k) is true (i.e., "$a_1 \cdots a_k \leq b_1 \cdots b_k$").
4. Inductive Step:

   Goal: show P(k+1), i.e., "$a_1 \cdots a_{k+1} \leq b_1 \cdots b_{k+1}$"

   From givens, we have $a_{k+1} \leq b_{k+1}$ (∀ Elim). Then,

$a_1 \cdots a_{k+1}$	$= a_1 \cdots a_k a_{k+1}$	show one more in "…"
	$\leq (b_1 \cdots b_k) a_{k+1}$	by IH
	$\leq b_1 \cdots b_k b_{k+1}$	by above

   Therefore P(k+1) is true.
5. Thus P(n) is true for all integers n ≥ 1, by induction.

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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2. **Base Case** (n=2):

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: for an arbitrary integer** $k \geq 2$, **suppose that P(k) is true (i.e., “$3^k \geq k^2+3$”).**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3$

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   suppose that P(k) is true (i.e., "$3^k \geq k^2+3$").
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

$3^k \geq 3$

$k^2 \geq 2k$

$3^k \geq k^2+3 \geq 2k+3$

$2 \cdot 3^k \geq k^2+2k+6$

$3 \cdot 3^k \geq k^2+2k+9 \geq k^2+2k+4.$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
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   suppose that P(k) is true (i.e., "$3^k \geq k^2+3$").
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

$3^{k+1} = 3(3^k)$

$\geq 3(k^2+3)$ **by the IH**

~~$= k^2+2k^2+9$~~ $= 3k^2 + 9 = k^2 + 2k^2 + 9$

$\geq k^2+2k+4 = (k+1)^2+3$ **since** $k \geq 1$.

$(\geq k)\cdot k$

$2k$

$\geq k^2 + 2k + 9$

$\geq k^2 + 2k + 4$.

Therefore P(k+1) **is true.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
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   Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

   $$3^{k+1} = 3(3^k)$$
   $$\geq 3(k^2+3) \text{ by the IH}$$
   $$= k^2+2k^2+9$$
   $$\geq k^2+2k+4 = (k+1)^2+3 \text{ since } k \geq 1.$$

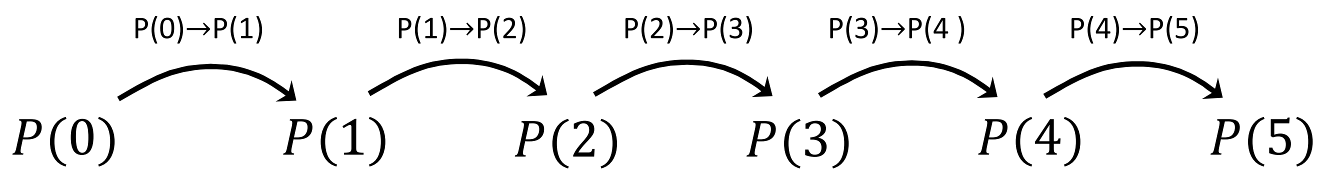
   Therefore P(k+1) **is true.**
5. **Thus** P(n) **is true for all integers** $n \geq 2$, **by induction.**

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

How do the givens prove P(5)?



P(0)

∀k P(k) → P(k+1)

P(0) → P(1)

P(1)

P(0) ∧ P(1) ∧ P(2) → P(3)

P(1) → P(2)

P(2)

P(2) → P(3)

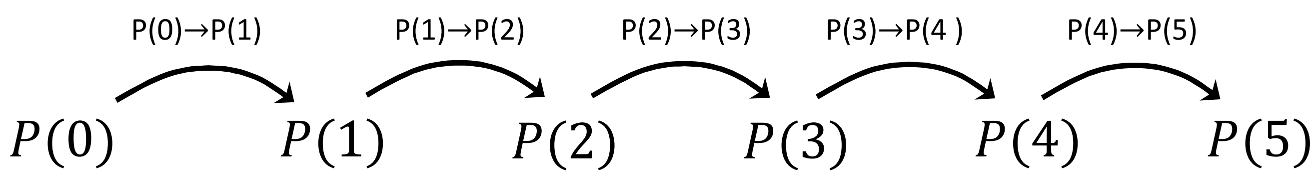
P(3)

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?



We made it harder than we needed to …

When we proved $P(2)$ **we knew BOTH** $P(0)$ **and** $P(1)$

When we proved $P(3)$ **we knew** $P(0)$ **and** $P(1)$ **and** $P(2)$

When we proved $P(4)$ **we knew** $P(0), P(1), P(2), P(3)$

etc.

That's the essence of the idea of Strong Induction.

Strong Induction

$$P(0)$$

$$\forall k \; \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k) \big) \rightarrow P(k+1) \Big)$$

$$\therefore \forall n \; P(n)$$

Strong Induction

Q(0)
∀k Q(k) → Q(k+1)

$$P(0)$$

$$\forall k\ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \rightarrow P(k+1)\Big)$$

$$\therefore \forall n\ P(n)$$

Strong induction for P follows from ordinary induction for Q where

$$Q(k) = P(0) \land P(1) \land P(2) \land \cdots \land P(k)$$

Note that $Q(0) \equiv P(0)$ and $Q(k+1) \equiv Q(k) \land P(k+1)$ and $\forall n\ Q(n) \equiv \forall n\ P(n)$

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(k)$ is true"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)

5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ *is true for every integer* j *from* b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

48 = 2 • 2 • 2 • 2 • 3
591 = 3 • 197
45,523 = 45,523
321,950 = 2 • 5 • 5 • 47 • 137
1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

We use strong induction to prove that a factorization into primes exists, but not that it is unique.

Every integer ≥ 2 is a product of primes.

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1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n $\geq$ 2 **by strong induction.**

Base Case ($n = 2$): 2 is prime, so its a trivial product of primes.

–

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) i**s true for all integers** n ≥ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a (trivial) product of primes. Therefore,** P(2) **is true.**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n ≥ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore, P(2) **is true.**
3. **Inductive Hyp: Suppose that, for an arbitrary integer** k ≥ 2,
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

If k+1 is prime, then P(k+1) is immediate.

Otherwise, $k+1 = ab$ for some $2 \le a, b < k+1$.

Write $a = p_1 \cdots p_j$ and $b = q_1 \cdots q_\ell$ for some primes.

So $k+1 = ab = p_1 \cdots p_j q_1 \cdots q_\ell$, which proves P(k+1).

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n $\geq$ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore, P(2) **is true.**
3. **Inductive Hyp: Suppose that, for an arbitrary integer** k ≥ 2**,**
   P(j) **is true for every integer** j **between** 2 **and** k
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   <u>**Case:** k+1 **is prime**</u>**: Then by definition** k+1 **is a product of primes**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n ≥ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes. Therefore,** P(2) **is true.**
3. **Inductive Hyp: Suppose that, for an arbitrary integer** k ≥ 2, P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

   Case: k+1 **is prime: Then by definition** k+1 **is a product of primes**

   Case: k+1 **is composite: Then** k+1=ab **for some integers** a **and** b **where** 2 ≤ a, b ≤ k.

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore, P(2) **is true.**
3. **Inductive Hyp: Suppose that, for an arbitrary integer** $k \geq 2$,
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

   <u>**Case:** k+1 **is prime**</u>: **Then by definition** k+1 **is a product of primes**

   <u>**Case:** k+1 **is composite:**</u> **Then** k+1=ab **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** P(a) **and** P(b) **are true so we have**

   $a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$

   for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

   Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore, P(2) **is true.**
3. **Inductive Hyp: Suppose that, for an arbitrary integer** $k \geq 2$,
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

   <u>Case: k+1 is prime</u>: Then by definition k+1 **is a product of primes**
   <u>Case: k+1 is composite:</u> Then k+1=ab **for some integers** a **and** b
   where $2 \leq a, b \leq k$. **By our IH,** P(a) **and** P(b) **are true so we have**
   $a = p_1p_2 \cdots p_r$ **and** $b = q_1q_2 \cdots q_s$
   for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.
   Thus, $k+1 = ab = p_1p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

   Since $k \geq 1$, **one of these cases must happen and so** P(k+1) **is true:**
5. **Thus** P(n) **is true for all integers** $n \geq 2$, **by strong induction.**

Strong Induction is particularly useful when…

…we need to analyze methods that on input k make a recursive call for an input different from $k-1$.

e.g.: Recursive Modular Exponentiation:

- For exponent $k > 0$ it made a recursive call with exponent $\mathrm{j} = k/2$ when k was even or $\mathrm{j} = k-1$ when k was odd.

We won't analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.

Recursive definitions of functions

- $F(0) = 0;\ \ F(n+1) = F(n) + 1$ **for all** $n \geq\ 0$.

$F(n) = n$

- $G(0) = 1;\ \ G(n+1) = 2 \cdot G(n)$ **for all** $n \geq\ 0$.

$G(n) = 2^n$

- $0! = 1;\ \ (n+1)! = (n+1) \cdot n!$ **for all** $n \geq\ 0$.

$n! = n \cdot (n-1) \cdots 2 \cdot 1$

- $H(0) = 1;\ \ H(n+1) = 2^{H(n)}$ **for all** $n \geq\ 0$.

$H(n) = 2^{2^{\cdot^{\cdot^{2}}}} \Big\} n$

Prove $n! \leq n^n$ for all $n \geq 1$

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "n! ≤ n^n". **We will show that** P(n) **is true for all integers** n ≥ 1 **by induction.**

2. Base Case (n=1). $n! = 1 \cdot 0! = 1 \cdot 1 = 1$

$1^1 = 1$

both sides are 1 so P(1) is true.

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "$n! \leq n^n$". **We will show that** P(n) **is true for all integers** $n \geq 1$ **by induction.**
2. **Base Case** (n=1): $1! = 1 \cdot 0! = 1 \cdot 1 = 1 = 1^1$ **so** P(1) **is true.**

Prove $n! \le n^n$ for all $n \ge 1$

1. **Let** P(n) **be** "n! ≤ n^n". **We will show that** P(n) **is true for all integers** n ≥ 1 **by induction.**
2. **Base Case** (n=1): $1!=1\cdot 0!=1\cdot 1=1=1^1$ **so** P(1) **is true.**
3. **Inductive Hypothesis: for an arbitrary k >=1,**
   suppose that P(k) **is true (i.e.,** "$k! \le k^k$"**).**
4. **Inductive Step:**

   Goal: Show P(k+1), **i.e. show** $(k+1)! \le (k+1)^{k+1}$

$$(k+1)! = (k+1)\cdot k! \quad \text{by def}$$

$$\le (k+1)\,k^k \quad \text{by IH}$$

$$\le (k+1)(k+1)^k \quad \text{by first ex}$$

$$a_k = k, \quad b_k = k+1$$

$$= (k+1)^{k+1}$$

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "$n! \leq n^n$". **We will show that** P(n) **is true for all integers** $n \geq 1$ **by induction.**
2. **Base Case** (n=1): $1!=1\cdot 0!=1\cdot 1=1=1^1$ **so** P(1) **is true.**
3. **Inductive Hypothesis: for an arbitrary k >=1, suppose that P(k) is true (i.e., "$k! \leq k^k$").**
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $(k+1)! \leq (k+1)^{k+1}$

$(k+1)! = (k+1)\cdot k!$	**by definition of !**
$\leq (k+1)\cdot k^k$	**by the IH**
$\leq (k+1)\cdot (k+1)^k$	**by first ex. &** $k \leq k+1$ **for all k**
$= (k+1)^{k+1}$	

Therefore P(k+1) **is true.**

Prove $n! \le n^n$ for all $n \ge 1$

1. **Let** P(n) **be** "n! ≤ n^n". **We will show that** P(n) **is true for all integers** n ≥ 1 **by induction.**
2. **Base Case** (n=1): 1!=1·0!=1·1=1=1^1 **so** P(1) **is true.**
3. **Inductive Hypothesis: for an arbitrary k >=1,**
   suppose that P(k) is true (i.e., "k! ≤ k^k").
4. **Inductive Step:**

   Goal: Show P(k+1), **i.e. show** (k+1)! ≤ $(k+1)^{k+1}$

   $(k+1)! = (k+1)\cdot k!$ **by definition of !**

   $\le (k+1)\cdot k^k$ **by the IH**

   $\le (k+1)\cdot (k+1)^k$ **by first ex. &** k ≤ k+1 **for all k**

   $= (k+1)^{k+1}$

   Therefore P(k+1) **is true.**

5. **Thus** P(n) **is true for all** n ≥ 1, **by induction.**