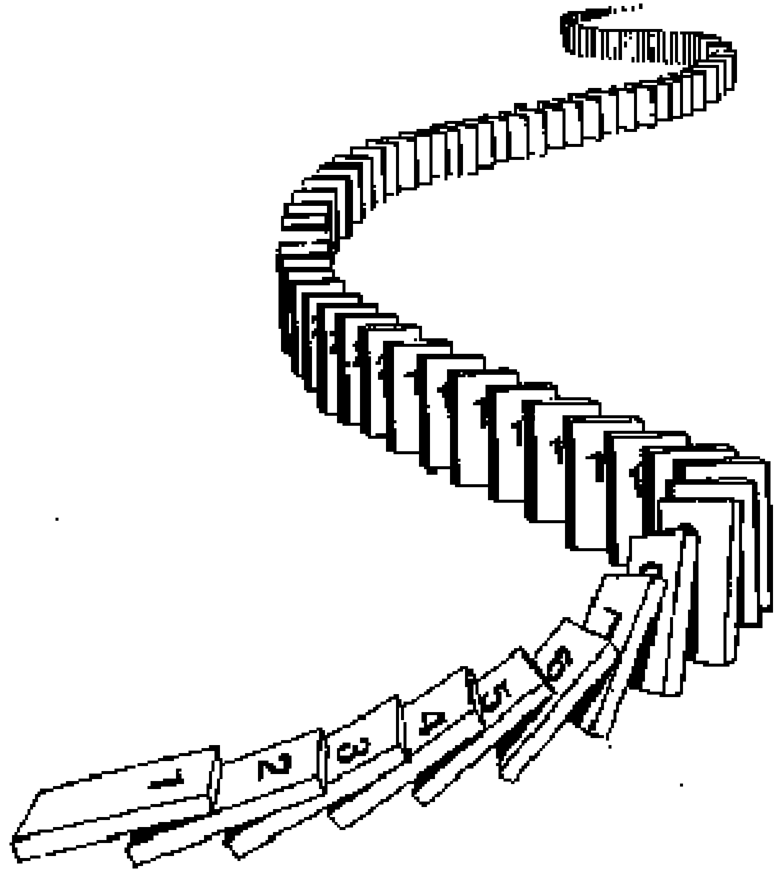
CSE 311: Foundations of Computing

Lecture 14: Induction

- Pick up copies of Homework 3 solutions if you haven't got them already.

HW4
- Boolean algebra identities apply to sets too. (But they don't help with subsets)



Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- **Particularly useful for reasoning about programs!**

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

```
public int f(int x) {
    if (x == 0) { return 0; }
    else { return f(x – 1); }
}
```

  - f(x) = x for all values of x ≥ 0 naturally shown by induction.

Prove $\forall a, b, m > 0\ \forall k \in \mathbb{N}\ (a \equiv b \pmod{m} \rightarrow a^k \equiv b^k \pmod{m})$

Let $a, b, m > 0 \in \mathbb{Z}$ **be arbitrary. Let** $k \in \mathbb{N}$ **be arbitrary. Suppose that** $a \equiv b \pmod{m}$**.**

We know $\big(a \equiv b \pmod{m} \wedge a \equiv b \pmod{m}\big) \rightarrow a^2 \equiv b^2 \pmod{m}$ **by multiplying congruences. So, applying this repeatedly, we have:**

$$\big(a \equiv b \pmod{m} \wedge a \equiv b \pmod{m}\big) \rightarrow a^2 \equiv b^2 \pmod{m}$$
$$\big(a^2 \equiv b^2 \pmod{m} \wedge a \equiv b \pmod{m}\big) \rightarrow a^3 \equiv b^3 \pmod{m}$$

...

$$\big(a^{k-1} \equiv b^{k-1} \pmod{m} \wedge a \equiv b \pmod{m}\big) \rightarrow a^k \equiv b^k \pmod{m}$$

The "…"s is a problem! We don't have a proof rule that allows us to say "do this over and over".

But there such a property of the natural numbers!

Domain: Natural Numbers

$$
\begin{array}{c}
P(0) \\
\forall k\ (P(k) \longrightarrow P(k+1)) \\
\hline
\therefore \forall n\ \ P(n)
\end{array}
$$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$
\begin{array}{rc}
1. & P(0) \\
2. & \forall k\ (P(k) \longrightarrow P(k+1)) \\
\hline
 & \therefore \forall n\ \ P(n)
\end{array}
$$

How do the givens prove P(5)?

1. $P(0)$: Given
3. $P(0) \to P(1)$ Elim $\forall$ from 2.
4. $P(1)$ MP: 1 & 3
5. $P(1) \to P(2)$ Elim ~~$\forall$ from~~ 2
6. $P(2)$ MP: 4 & 5.

$P(2) \to P(3)$

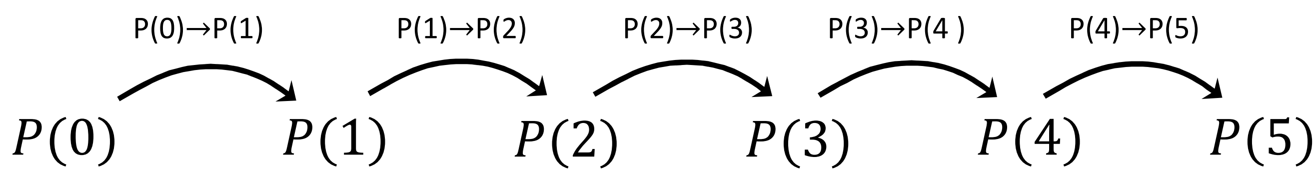
$P(0) \to P(1) \to P(2) \to P(3)$ - . .

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\frac{P(0) \qquad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n\;\; P(n)}$$

How do the givens prove P(5)?



First, we have P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Using The Induction Rule In A Formal Proof

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

Prove
1. P(0)
2. Let a be arbitrary natural #
3.1 P(a) Assumption
⋮
3.5 P(a+1)
3. P(a) → P(a+1)
4. ∀k (P(k) → P(k+1)) Intro ∀ : 2 -> 4
5. ∀n P(n) Ind Rule: from 1 & 4

Using The Induction Rule In A Formal Proof

$$\frac{P(0) \qquad \forall k\ (P(k) \longrightarrow P(k+1))}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)

4. $\forall$k (P(k) $\rightarrow$ P(k+1))
5. $\forall$n P(n) Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0

3. $P(k) \rightarrow P(k+1)$
4. $\forall k\ (P(k) \rightarrow P(k+1))$ Intro ∀: 2, 3
5. $\forall n\ P(n)$ Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{P(0) \qquad \forall k\, (P(k) \longrightarrow P(k+1))}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
   - 3.1. Assume that P(k) is true
   - 3.2. ...
   - 3.3. Prove P(k+1) is true
3. $P(k) \rightarrow P(k+1)$ — Direct Proof Rule
4. $\forall k\, (P(k) \rightarrow P(k+1))$ — Intro ∀: 2, 3
5. $\forall n\, P(n)$ — Induction: 1, 4

Translating to an English Proof

$$\frac{P(0) \qquad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore\ \forall n\ \ P(n)}$$

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0
 3.1. Assume that P(k) is true

Inductive Hypothesis

 3.2. ...
 3.3. Prove P(k+1) is true

Inductive Step

3.	P(k) → P(k+1)	Direct Proof Rule
4.	∀k (P(k) → P(k+1))	Intro ∀: 2, 3
5.	∀n P(n)	Induction: 1, 4

Conclusion

Translating To An English Proof

1. Prove P(0)		**Base Case**
2. Let k be an arbitrary integer ≥ 0		**Inductive Hypothesis**
3.1. Assume that P(k) is true		
3.2. ...		**Inductive Step**
3.3. Prove P(k+1) is true		
3. $P(k) \rightarrow P(k+1)$	Direct Proof Rule	**Conclusion**
4. $\forall k\,(P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3	
5. $\forall n\, P(n)$	Induction: 1, 4	

Induction Proof Template

[...Define P(n)...]

We will show that $P(n)$ is true for every $n \in \mathbb{N}$ by Induction.

Base Case: *[...proof of $P(0)$ here...]*

Induction Hypothesis:

Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step:

We want to prove that $P(k+1)$ is true.

[...proof of $P(k+1)$ here...]

*The proof of $P(k+1)$ **must** invoke the IH somewhere.*

So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Proof:

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 \quad = 1$
- $1 + 2 \quad = 3$
- $1 + 2 + 4 \quad = 7$
- $1 + 2 + 4 + 8 \quad = 15$
- $1 + 2 + 4 + 8 + 16 \quad = 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.

Good that we have induction!

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Proof: 1. Let P(n) be

"$1+2+\cdots+2^n = 2^{n+1}-1$"

we will prove P(n) for all $n \geq 0$ by induction

2. Base Case: n=0: $1+\cdots+2^0 = 1$ LS

$2^{0+1}-1 = 2-1 = 1$ RS = ✓

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + 2^n = 2^{n+1} – 1". **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**

3. Induction Hypothesis: Assume that P(k) is true for some (arbitrary) integer $k \geq 0$. (i.e. $1+2+\cdots+2^k = 2^{k+1} - 1$)

4. Inductive Step:

Goal: Show P(k+1) is true

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

By I.H. $1 + 2 + \cdots + 2^k = 2^{k+1} - 1$

$\therefore (1 + 2 + \cdots + 2^k) + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$ by IH

$= 2 \cdot (2^{k+1}) - 1$

$= 2^{k+2} - 1$ ✓

$\therefore$ P(k+1) is true

5. Conclusion: By induction we have shown P(n) is true for all integers $n \geq 0$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

   $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ **by IH**

   Adding 2^{k+1} **to both sides, we get:**

   $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** P(k+1).

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** $P(n)$ **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** $P(0)$ **is true.**
3. **Induction Hypothesis: Suppose that** $P(k)$ **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

Goal: Show $P(k+1)$, **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

$$1 + 2 + \ldots + 2^k + 2^{k+1} = (1+2+ \ldots + 2^k) + 2^{k+1}$$
$$= 2^{k+1} - 1 + 2^{k+1} \text{ by the IH}$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** $P(k+1)$.

Alternative way of writing the inductive step

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**.**
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

$$1 + 2 + \ldots + 2^k + 2^{k+1} = (1+2+ \ldots + 2^k) + 2^{k+1}$$
$$= 2^{k+1} - 1 + 2^{k+1} \text{ by the IH}$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$**.**

So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** P(k+1)**.**

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$, **by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Proof

1. Let $P(n)$ be "$1+2+\cdots+n = n(n+1)/2$"

2. Base Case: $P(0)$. LHS = 0 RHS = 0 ✓

3. IH. Assume that $P(k)$ is true for some arbitrary integer $k \geq 0$

4. I.S.

Goal Show $P(k+1)$

$1+2+\cdots+k+(k+1) = (k+1)(k+2)/2$

$$1+2+3+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) \text{ by IH.}$$

$$= (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}$$

5. Conclusion

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**.**
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** 1 + 2 + …+ k+ (k+1) = (k+1)(k+2)/2

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**.**
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** 1 + 2 + …+ k+ (k+1) = (k+1)(k+2)/2

$$1 + 2 + \ldots + k + (k+1) = (1 + 2 + \ldots + k) + (k+1)$$

$$= k(k+1)/2 + (k+1) \quad \textbf{by IH}$$

Now k(k+1)/2 + (k+1) = (k+1)(k/2 + 1) = (k+1)(k+2)/2**.**

So, we have 1 + 2 + … + k + (k+1) = (k+1)(k+2)/2**, which is exactly** P(k+1)**.**

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

Another example of a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- …

$3 \mid (2^{2n} - 1)$ $n \geq 0$

Prove: $3 \mid (2^{2n}-1)$ **for all** $n \geq 0$

Let P(n) be "$3 \mid (2^{2n}-1)$"

We ~~prove~~ P(n) ~~for~~ all $n \geq 0$ by induction

Prove: $3 \mid (2^{2n} - 1)$ **for all** $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2 \cdot 0} - 1 = 2^0 - 1 = 0$ $3 \mid 0$ ✓

Prove: $3 \mid (2^{2n} - 1)$ for all $n \geq 0$

1. **Let** P(n) **be** "3 | (2^{2n} – 1)". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}$-1=1-1=0=3·0 **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**.**
4. **Induction Step:**

Goal: Show P(k+1)**, i.e. show** 3 | ($2^{2(k+1)}$ – 1)

By IH $3 \mid (2^{2k} - 1)$

Prove: $3 \mid (2^{2n}-1)$ for all $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}-1=1-1=0=3\cdot 0$ **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**.**
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** $3 \mid (2^{2(k+1)} - 1)$

By IH, $3 \mid (2^{2k} - 1)$ **so** $2^{2k} - 1 = 3j$ **for some integer** j

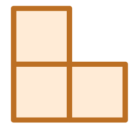
So $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 4(2^{2k}) - 1 = 4(3j+1) - 1$

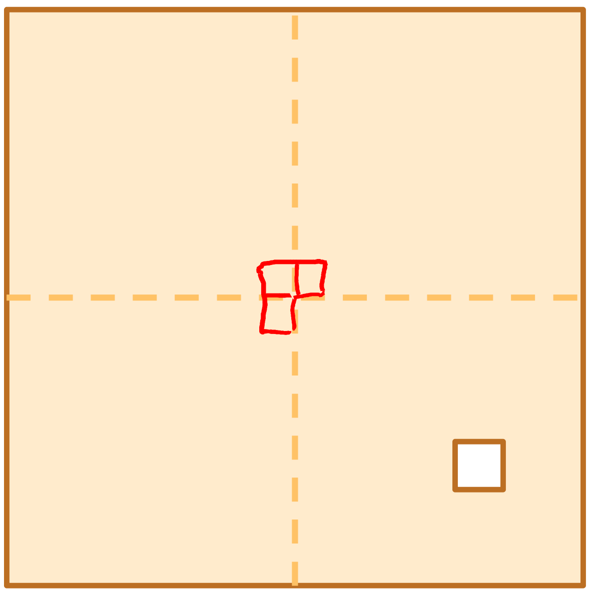
$= 12j+3 = 3(4j+1)$

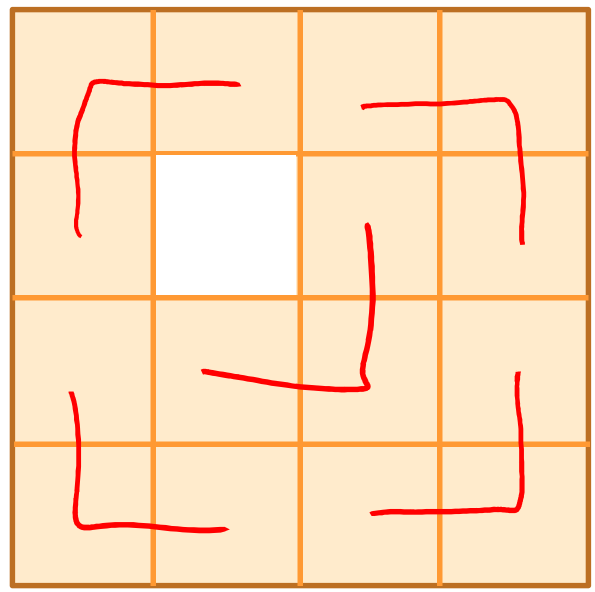
Therefore $3 \mid (2^{2(k+1)} - 1)$ **which is exactly** P(k+1)**.**

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

Checkerboard Tiling

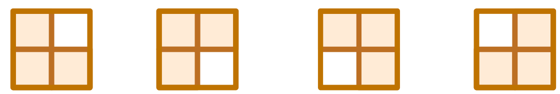
- Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 



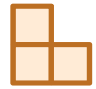


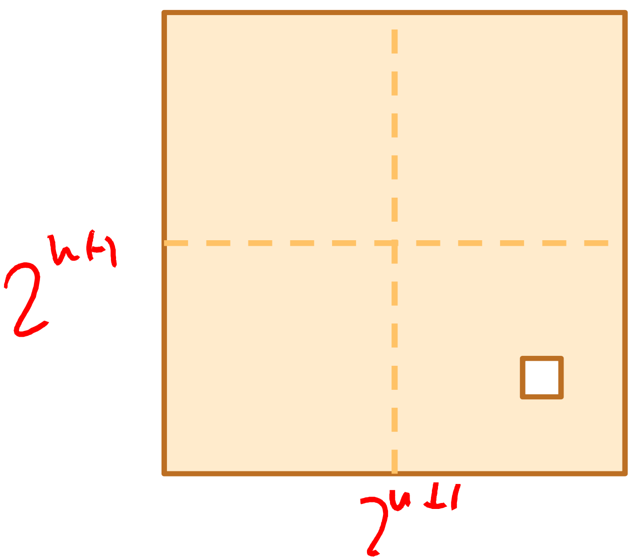
Checkerboard Tiling

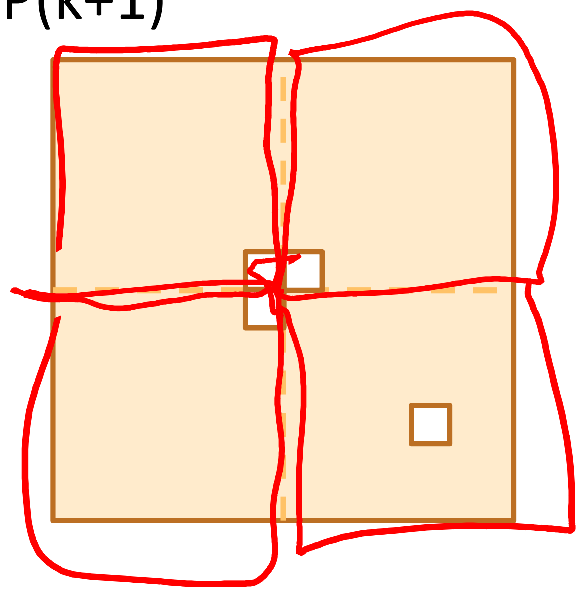
1. **Let** $P(n)$ **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove $P(n)$ **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** $n=1$



Checkerboard Tiling

1. Let P(n) be any $2^n \times 2^n$ checkerboard with one square removed can be tiled with  .
   We prove P(n) for all $n \geq 1$ by induction on n.
2. Base Case: n=1 
3. Inductive Hypothesis: Assume P(k) for some arbitrary integer $k \geq 1$
4. Inductive Step: Prove P(k+1)





Apply IH to each quadrant then fill with extra tile.