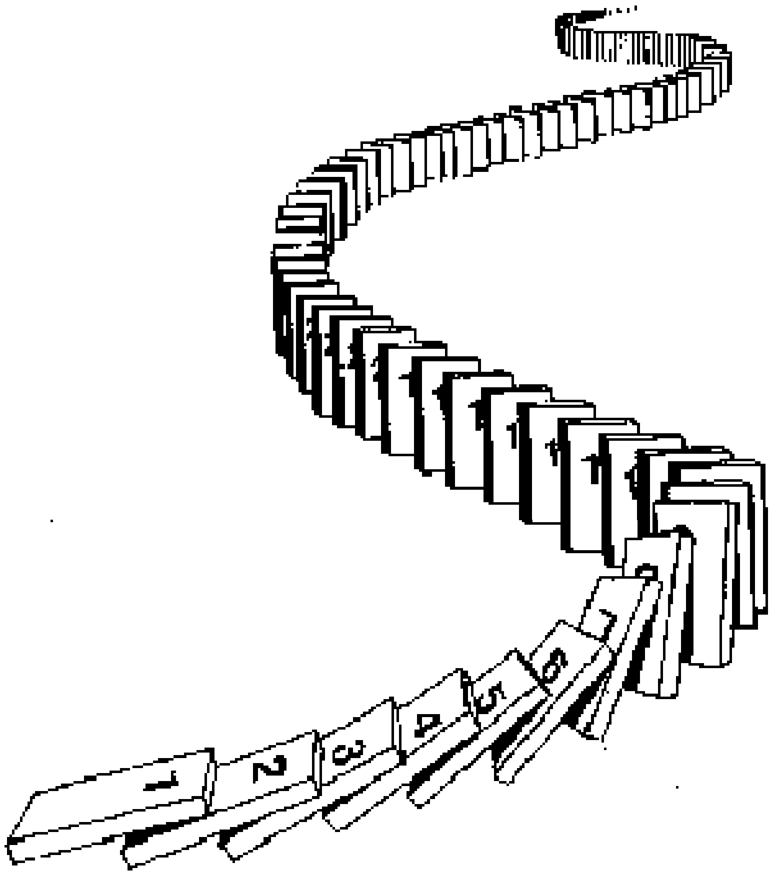
CSE 311: Foundations of Computing

Lecture 14: Induction



Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- **Particularly useful for reasoning about programs!**

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

```
public int f(int x) {  /* x >= 0 */
    if (x == 0) { return 0; }
    else { return f(x – 1) + 1; }
}
```

  - f(x) = x for all values of x ≥ 0 naturally shown by induction.

Prove $\forall a, b, m > 0 \; \forall k \in \mathbb{N} \; (a \equiv b \pmod{m} \rightarrow a^k \equiv b^k \pmod{m})$

Let $a, b, m > 0 \in \mathbb{Z}$ **be arbitrary. Let** $k \in \mathbb{N}$ **be arbitrary. Suppose that** $a \equiv b \pmod{m}$**.**

We know $\big(a \equiv b \pmod{m} \wedge a \equiv b \pmod{m}\big) \rightarrow a^2 \equiv b^2 \pmod{m}$ **by multiplying congruences. So, applying this repeatedly, we have:**

$$\big(a \equiv b \pmod{m} \wedge a \equiv b \pmod{m}\big) \rightarrow a^2 \equiv b^2 \pmod{m}$$
$$\big(a^2 \equiv b^2 \pmod{m} \wedge a \equiv b \pmod{m}\big) \rightarrow a^3 \equiv b^3 \pmod{m}$$
$$\ldots$$
$$\left(a^{k-1} \equiv b^{k-1} \pmod{m} \wedge a \equiv b \pmod{m}\right) \rightarrow a^k \equiv b^k \pmod{m}$$

The "…"s is a problem! We don't have a proof rule that allows us to say "do this over and over".

But there such a property of the natural numbers!

Domain: Natural Numbers

$\in \mathbb{N}$

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\overline{\therefore \forall n\ \ P(n)}$$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

1. $P(0)$ Given
2. $\forall k\,(P(k) \to P(k+1))$ "
3. $P(0) \to P(1)$ $\forall$ Elim
4. $P(1)$ MP
5. $P(1) \to P(2)$
6. $P(2)$
7. $P(2) \to P(3)$
8. $P(3)$
9. $P(3) \to P(4)$
10. $P(4)$
11. $P(4) \to P(5)$
12. $P(5)$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$P(0) \xrightarrow{P(0)\to P(1)} P(1) \xrightarrow{P(1)\to P(2)} P(2) \xrightarrow{P(2)\to P(3)} P(3) \xrightarrow{P(3)\to P(4)} P(4) \xrightarrow{P(4)\to P(5)} P(5)$$

First, we have P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
    Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
    Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Using The Induction Rule In A Formal Proof

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

5. $\forall n\ P(n)$ Induction:

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)

4. $\forall k\ (P(k) \rightarrow P(k+1))$
5. $\forall n\ P(n)$ Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0

3.	$P(k) \rightarrow P(k+1)$	
4.	$\forall k\ (P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5.	$\forall n\ P(n)$	Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{P(0) \qquad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n\ P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
   3.1. Assume that P(k) is true
   3.2. ...
   3.3. Prove P(k+1) is true
3. $P(k) \rightarrow P(k+1)$ — Direct Proof Rule
4. $\forall k\,(P(k) \rightarrow P(k+1))$ — Intro $\forall$: 2, 3
5. $\forall n\, P(n)$ — Induction: 1, 4

Translating to an English Proof

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0
    3.1. Assume that P(k) is true

Inductive Hypothesis

    3.2. ...
    3.3. Prove P(k+1) is true

Inductive Step

3.	$P(k) \rightarrow P(k+1)$	Direct Proof Rule
4.	$\forall k\ (P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5.	$\forall n\ P(n)$	Induction: 1, 4

Conclusion

Translating To An English Proof

1. Prove P(0)		**Base Case**
2. Let k be an arbitrary integer ≥ 0		**Inductive Hypothesis**
3.1. Assume that P(k) is true		
3.2. ...		**Inductive Step**
3.3. Prove P(k+1) is true		
3. P(k) → P(k+1)	Direct Proof Rule	
4. ∀k (P(k) → P(k+1))	Intro ∀: 2, 3	
5. ∀n P(n)	Induction: 1, 4	**Conclusion**

Induction Proof Template

[...Define P(n)...]

We will show that $P(n)$ is true for every $n \in \mathbb{N}$ by Induction.

Base Case: *[...proof of $P(0)$ here...]*

Induction Hypothesis:

Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step:

We want to prove that $P(k+1)$ is true.

[...proof of $P(k+1)$ here...]

*The proof of $P(k+1)$ **must** invoke the IH somewhere.*

So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Proof:

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 \qquad = \quad 1$
- $1 + 2 \qquad = \quad 3$ $\quad 4 - 1$
- $1 + 2 + 4 \qquad = \quad 7$ $\quad 8 - 1$
- $1 + 2 + 4 + 8 \qquad = \quad 15$ $\quad 16 - 1$
- $1 + 2 + 4 + 8 + \underbrace{16}_{2^n} \qquad = \quad 31$ $\quad 32 - 1$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.

Good that we have induction!

Prove $1 + 2 + 4 + ... + 2^n = 2^{n+1} - 1$

Prove $1 + 2^1 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + 2^n = 2^{n+1} – 1". **We will show** P(n) **is true for all natural numbers by induction.**

Base Case: (n=0): $2^0 = 1$

LHS = 1 RHS: $2^{0+1} - 1 = 2^1 - 1$

$= 2 - 1 = 1$

These are equal, so P(0) is true.

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. Let $P(n)$ be "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". We will show $P(n)$ is true for all natural numbers by induction.
2. Base Case $(n=0)$: $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so $P(0)$ is true.
3. Induction Hypothesis: for an arbitrary integer $k \geq 0$, suppose that $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ $P(k)$

Induction Step: Want to prove $P(k+1)$

$$1 + 2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$$

$$\rightarrow 1 + 2 + \cdots + 2^k + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^{k+1} \quad \text{by IH}$$

$$= 2(2^{k+1}) - 1$$

$$\rightarrow = 2^{k+2} - 1$$

which shows $P(k+1)$ is true.

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: for an arbitrary integer $k \geq 0$, suppose that $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$**
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** $P(n)$ **is true for all natural numbers by induction.**
2. **Base Case** $(n=0)$: $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** $P(0)$ **is true.**
3. **Induction Hypothesis: for an arbitrary integer $k \geq 0$, suppose that $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$**
4. **Induction Step:**

$$\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1}$$

$1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ **by IH**

Adding 2^{k+1} **to both sides, we get:**

$1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$**.**

So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$**, which is exactly** $P(k+1)$**.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: for an arbitrary integer k ≥ 0, suppose that $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$**
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

$$1 + 2 + \ldots + 2^k + 2^{k+1} = (1+2+ \ldots + 2^k) + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^{k+1} \text{ by the IH}$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$**.**

So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** P(k+1).

Alternative way of writing the inductive step

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: for an arbitrary integer k ≥ 0, suppose that $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$**
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

$$1 + 2 + \ldots + 2^k + 2^{k+1} = (1+2+ \ldots + 2^k) + 2^{k+1}$$
$$= 2^{k+1} - 1 + 2^{k+1} \text{ by the IH}$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** P(k+1).

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$, **by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

Base Case (n=0)   LHS = 0

RHS: $0 \cdot (0+1)/2 = 0$

These are equal, so P(0) is true.

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: for an arbitrary integer k ≥ 0, suppose that** 1 + 2 + …+ k = k(k+1)/2
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** 1 + 2 + …+ k+ (k+1) = (k+1)(k+2)/2

$$\Rightarrow (1 + 2 + \cdots + k) + (k+1)$$
$$= k(k+1)/2 + (k+1) \qquad \text{by IH}$$
$$= (k+1)(k/2 + 1)$$
$$= (k+1)(k/2 + 2/2)$$
$$= (k+1)(k+2)/2$$

which shows P(k+1).

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: for an arbitrary integer** k ≥ 0, **suppose that** 1 + 2 + …+ k = k(k+1)/2
4. **Induction Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 1 + 2 + \ldots + k + (k+1) = (k+1)(k+2)/2}$

   $$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= (1 + 2 + \ldots + k) + (k+1) \\ &= k(k+1)/2 + (k+1) \quad \textbf{by IH} \end{aligned}$$

   Now k(k+1)/2 + (k+1) = (k+1)(k/2 + 1) = (k+1)(k+2)/2**.**

   So, we have 1 + 2 + … + k + (k+1) = (k+1)(k+2)/2**, which is exactly** P(k+1)**.**
5. **Thus** P(n) **is true for all** n ∈ℕ**, by induction.**

Another example of a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- …

Prove: $3 \mid (2^{2n} - 1)$ **for all** $n \geq 0$

Prove: $3 \mid (2^{2n} - 1)$ for all $n \geq 0$

1. Let P(n) be "3 | ($2^{2n}-1$)". We will show P(n) is true for all natural numbers by induction.
2. Base Case (n=0):

$2^{2\cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$

and $3 \mid 0$ so P(0) is true

Prove: $3 \mid (2^{2n} - 1)$ for all $n \geq 0$

1. **Let** P(n) **be** "3 | (2^{2n} – 1)". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}-1=1-1=0=3\cdot 0$ **Therefore** P(0) **is true.**
3. **Induction Hypothesis: for an arbitrary integer** k ≥ 0, **suppose that** 3 | (2^{2k} – 1).
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** 3 | ($2^{2(k+1)}$ – 1)

IH says $2^{2k} - 1 = 3j$ for some $j \in \mathbb{Z}$.

Thus, $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 4 \cdot 2^{2k} - 1$

$= 4(3j+1) - 1$ by IH.

$= 12j + 4 - 1 = 12j + 3$

$= 3(4j+1)$, so $3 \mid 2^{2(k+1)} - 1$

Prove: $3 \mid (2^{2n} - 1)$ for all $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}-1=1-1=0=3\cdot 0$ **Therefore** P(0) **is true.**
3. **Induction Hypothesis: for an arbitrary integer k ≥ 0, suppose that** $3 \mid (2^{2k} - 1)$.
4. **Induction Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 3 \mid (2^{2(k+1)} - 1)}$

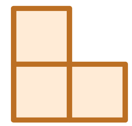
   By IH, $3 \mid (2^{2k} - 1)$ **so** $2^{2k} - 1 = 3j$ **for some integer** j

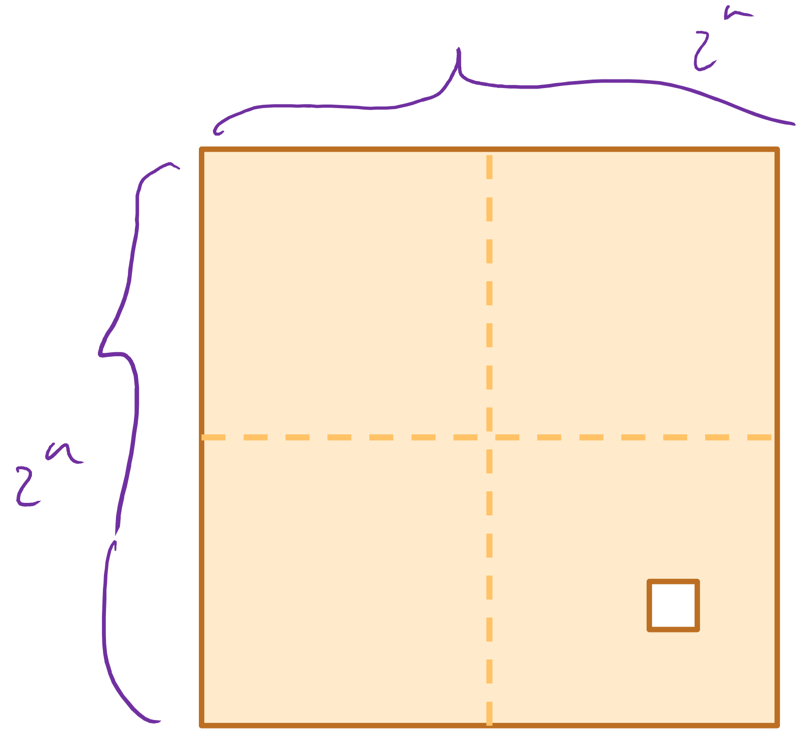
   So $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 4(2^{2k}) - 1 = 4(3j+1) - 1$

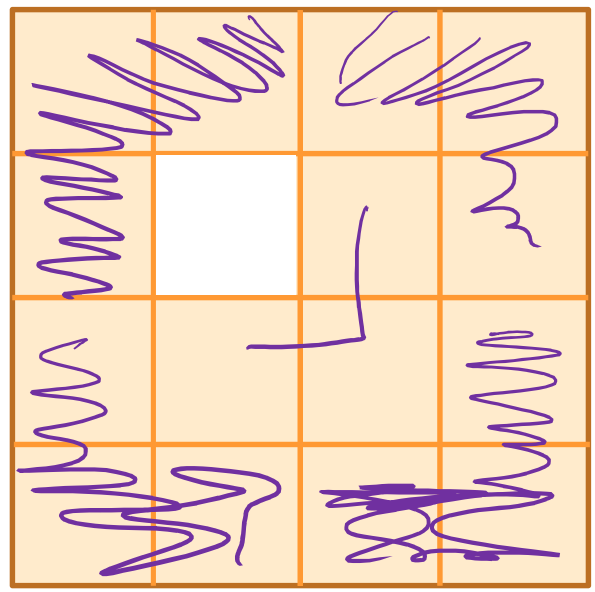
   $= 12j+3 = 3(4j+1)$

   Therefore $3 \mid (2^{2(k+1)} - 1)$ **which is exactly** P(k+1).
5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

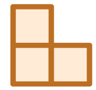
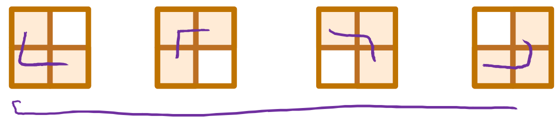
Checkerboard Tiling

- Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 

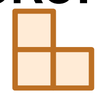
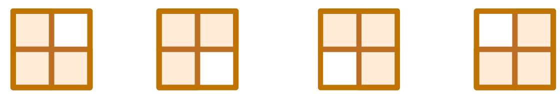




Checkerboard Tiling

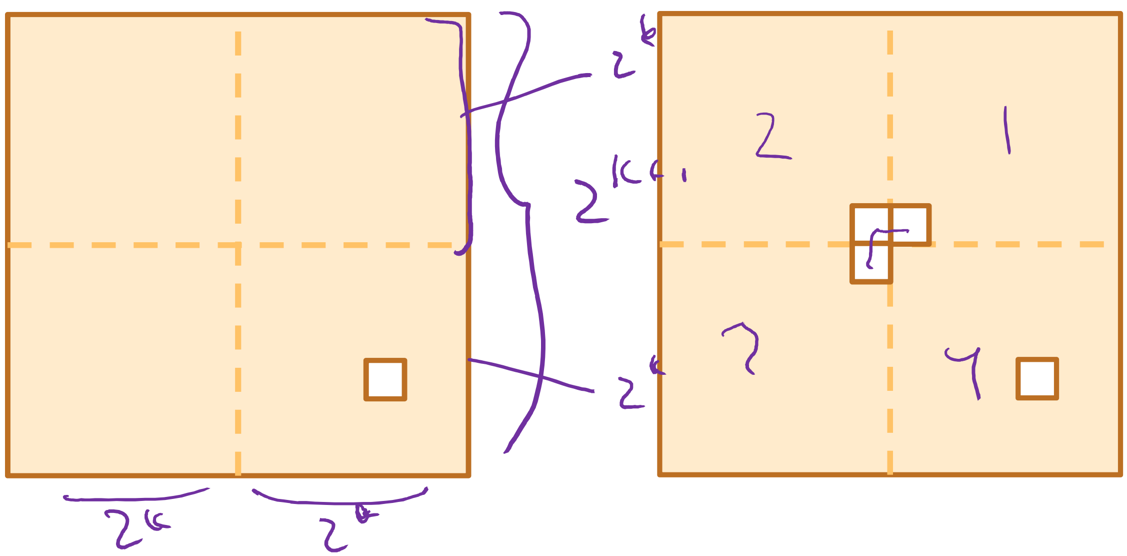
1. **Let** P(n) **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove P(n) **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** n=1 

Checkerboard Tiling

1. **Let** P(n) **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove P(n) **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** n=1 
3. **Inductive Hypothesis: Assume** P(k) **for some arbitrary integer** $k \geq 1$

holds for $2^k \times 2^k$ boards.

4. **Inductive Step: Prove** P(k+1)



Apply IH to each quadrant then fill with extra tile.