Lecture 8: Predicate Logic Proofs

The Axiom of Choice allows you to select one element from each set in a collection, and have it executed as an example to the others.

My Math Teacher was a big believer in proof by intimidation.
Last class: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it:

- Elimination (Elim) of conjunction ($\land$):
  \[ \frac{A \land B}{A, B} \]

- Introduction (Intro) of conjunction ($\land$):
  \[ \frac{A \land B}{\therefore A, B} \]

- Elimination (Elim) of disjunction ($\lor$):
  \[ \frac{A \lor B ; \neg A}{\therefore B} \]

- Introduction (Intro) of disjunction ($\lor$):
  \[ \frac{A}{\therefore A \lor B, B \lor A} \]

- Modus Ponens:
  \[ \frac{A ; A \rightarrow B}{\therefore B} \]

- Direct Proof Rule:
  \[ \frac{A \Rightarrow B}{\therefore A \rightarrow B} \]

Not like other rules.
Last class: Example

Prove: \(( (p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r) \)

1.1. \((p \rightarrow q) \land (q \rightarrow r)\) Assumption

1.2. \(p \rightarrow q\) \land Elim: 1.1

1.3. \(q \rightarrow r\) \land Elim: 1.1

1.4.1. \(p\) Assumption

1.4.2. \(q\) MP: 1.2, 1.4.1

1.4.3. \(r\) MP: 1.3, 1.4.2

1.4. \(p \rightarrow r\) Direct Proof Rule

1. \(( (p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r) \) Direct Proof Rule
One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given.

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.
**Inference Rules for Quantifiers: First look**

**Elim ∃**: \( \exists x \ P(x) \)

\[ \therefore \ P(c) \text{ for some special** c} \]

**Intro ∃**: \( P(c) \text{ for some c} \)

\[ \therefore \ \exists x \ P(x) \]

**Intro ∀**: \( \forall x \ P(x) \)

\[ \therefore \ P(a) \text{ for any a} \]

**Elim ∀**: \( \forall x \ P(x) \)

\[ \therefore \ \exists x \ P(x) \]

**“Let a be arbitrary*”**... \( P(a) \)

\[ \therefore \ \forall x \ P(x) \]

* in the domain of P

**By special, we mean that c is a name for a value where P(c) is true. We can’t use anything else about that value, so c has to be a NEW name!**
Predicate Logic Proofs

• Can use
  – Predicate logic inference rules
    whole formulas only
  – Predicate logic equivalences (De Morgan’s)
    even on subformulas
  – Propositional logic inference rules
    whole formulas only
  – Propositional logic equivalences
    even on subformulas
My First Predicate Logic Proof

Prove \((\forall x \ P(x)) \rightarrow (\exists x \ P(x))\)

The main connective is implication so Direct Proof Rule seems good
My First Predicate Logic Proof

Prove $\forall x \ P(x) \rightarrow \exists x \ P(x)$

1. $\forall x \ P(x)$ \hspace{1cm} Assumption

1.5. $\exists x \ P(x)$ \hspace{1cm} Direct Proof Rule

1. $\forall x \ P(x) \rightarrow \exists x \ P(x)$
My First Predicate Logic Proof

Prove $\forall x P(x) \rightarrow \exists x P(x)$

1. $\forall x P(x)$ Assumption

We need an $\exists$ we don’t have so “intro $\exists$” rule makes sense

1.5. $\exists x P(x)$

1. $\forall x P(x) \rightarrow \exists x P(x)$ Direct Proof Rule
My First Predicate Logic Proof

Prove \( \forall x \, P(x) \rightarrow \exists x \, P(x) \)

1. \( \forall x \, P(x) \rightarrow \exists x \, P(x) \)  \hspace{2cm} \text{Direct Proof Rule}

1.1. \( \forall x \, P(x) \)  \hspace{2cm} \text{Assumption}

1.5. \( \exists x \, P(x) \)  \hspace{2cm} \text{Intro } \exists : ?

We need an \( \exists \) we don’t have so “intro \( \exists \)” rule makes sense

That requires \( P(c) \) for some \( c \).
My First Predicate Logic Proof

Prove $\forall x \ P(x) \rightarrow \exists x \ P(x)$

1. $\forall x \ P(x)$  
   Assumption

2. $P(a)$  
   Elim $\forall$: 1.1

We could have picked any name or domain expression here.

1.5. $\exists x \ P(x)$  
   Intro $\exists$ for some c

1. $\forall x \ P(x) \rightarrow \exists x \ P(x)$  
   Direct Proof Rule
My First Predicate Logic Proof

Prove \( \forall x \ P(x) \rightarrow \exists x \ P(x) \)

1. \( \forall x \ P(x) \) Assumption
2. \( P(a) \) Elim \( \forall \): 1.1
5. \( \exists x \ P(x) \) Intro \( \exists \): 1.2

No holes. Just need to clean up.

1. \( \forall x \ P(x) \rightarrow \exists x \ P(x) \) Direct Proof Rule
My First Predicate Logic Proof

Prove $\forall x P(x) \rightarrow \exists x P(x)$

1. $\forall x P(x)$ Assumption
2. $P(a)$ Elim $\forall$: 1.1
3. $\exists x P(x)$ Intro $\exists$: 1.2

1. $\forall x P(x) \rightarrow \exists x P(x)$ Direct Proof Rule

Working forwards as well as backwards:
In applying “Intro $\exists$” rule we didn’t know what expression we might be able to prove $P(c)$ for, so we worked forwards to figure out what might work.
Predicate Logic Proofs with more content

• In propositional logic we could just write down other propositional logic statements as “givens”

• Here, we also want to be able to use domain knowledge so proofs are about something specific

• Example:

<table>
<thead>
<tr>
<th>Domain of Discourse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integers</td>
</tr>
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</table>

• Given the basic properties of arithmetic on integers, define:

<table>
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<tr>
<td>Even(x) \equiv \exists y (x = 2 \cdot y)</td>
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<td>Odd(x) \equiv \exists y (x = 2 \cdot y + 1)</td>
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Predicate Definitions

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</table>
| **Predicate Definitions** | Even(x) ≡ ∃y (x = 2·y)  
                          Odd(x) ≡ ∃y (x = 2·y + 1) |

Prove “There is an even number”
Formally: prove ∃x Even(x)

1. 0 = 2 · 0
2. ∃y (0 = 2·y)
3. Even(0)
4. ∃x Even(x)  
   ∃ Intro: 1  
   Defn of Even  

Arithmetic  

## A Not so Odd Example

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Prove “There is an even number”

Formally: prove $\exists x \text{ Even}(x)$

1. $2 = 2 \cdot 1$  
   *Arithmetic*
2. $\exists y (2 = 2 \cdot y)$  
   *Intro $\exists$: 1*
3. $\text{Even}(2)$  
   *Definition of Even: 2*
4. $\exists x \text{ Even}(x)$  
   *Intro $\exists$: 3*
## A Prime Example

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Prove “There is an even prime number”
A Prime Example

Predicate Definitions

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<td>Prime(x) ( \equiv ) “x &gt; 1 and x( \not= )a \cdot b for all integers a, b with 1&lt;a&lt;x”</td>
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Prove “There is an even prime number”
Formally: prove \( \exists x \ (\text{Even}(x) \land \text{Prime}(x)) \)

1. \( 2 = 2 \cdot 1 \)  
   \( \leftarrow \)  
   \( \text{Arithmetic} \)

2. \( \text{Prime}(2) \)  
   \( \leftarrow \)  
   \( \text{Property of integers} \)

3. \( \exists y \ (2 = 2 \cdot y) \)
   \( \text{Intro } \exists \ 1 \)

4. \( \text{Even}(2) \)
   \( \text{Def of Even } \text{: } 3 \)

5. \( \text{Even}(2) \land \text{Prime}(2) \)
   \( \text{Intro } \land \ 2, 4 \)

6. \( \exists x \ (\text{Even}(x) \land \text{Prime}(x)) \)
   \( \text{Intro } \exists \ 5 \)

* Later we will further break down “Prime” using quantifiers to prove statements like this
A Prime Example

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Even(x) \iff \exists y \ (x = 2 \cdot y)
Odd(x) \iff \exists y \ (x = 2 \cdot y + 1)
Prime(x) \iff “x > 1 and x \neq a \cdot b for all integers a, b with 1 < a < x”

Prove “There is an even prime number”
Formally: prove $\exists x \ (\text{Even}(x) \land \text{Prime}(x))$

1. $2 = 2 \cdot 1$                     Arithmetic
2. Prime(2) *                          Property of integers
3. $\exists y \ (2 = 2 \cdot y)$       Intro $\exists$: 1
4. Even(2)                             Defn of Even: 3
5. Even(2) $\land$ Prime(2)           Intro $\land$: 2, 4
6. $\exists x \ (\text{Even}(x) \land \text{Prime}(x))$ Intro $\exists$: 5

* Later we will further break down “Prime” using quantifiers to prove statements like this
Inference Rules for Quantifiers: First look

\[ \exists x \, P(x) \]

\[ \therefore P(c) \text{ for some special } c \]

** By special, we mean that c is a name for a value where P(c) is true. We can’t use anything else about that value, so c has to be a NEW name!

\[ \forall x \, P(x) \]

\[ \therefore P(a) \text{ for any } a \]

\[ \exists x \, P(x) \]

\[ \therefore \exists x \, P(x) \]

\[ \forall x \, P(x) \]

\[ \therefore \forall x \, P(x) \]

“Let a be arbitrary”...P(a)

* in the domain of P
Even and Odd

Prove: “The square of every even number is even.”

Formal proof of: \( \forall x \ (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)

3. \( \forall x \ (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)
Even and Odd

Prove: “The square of every even number is even.”

Formal proof of: \( \forall x \ (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)

1. Let \( a \) be an arbitrary integer

2. \( \text{Even}(a) \rightarrow \text{Even}(a^2) \)

3. \( \forall x \ (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)

Intro \( \forall \): 1, 2
Even and Odd

Prove: “The square of every even number is even.”

Formal proof of: \( \forall x \ (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)

1. Let \( a \) be an arbitrary integer
   2.1. \( \text{Even}(a) \) \hspace{1cm} Assumption

   \[ 2.6 \quad \text{Even}(a^2) \]

2. \( \text{Even}(a) \rightarrow \text{Even}(a^2) \)

3. \( \forall x \ (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)

Intro \( \forall \): 1,2

Even(x) \( \equiv \exists y \ (x=2y) \)
Odd(x) \( \equiv \exists y \ (x=2y+1) \)
Domain: Integers
Even and Odd

Prove: “The square of every even number is even.”

Formal proof of: \( \forall x \, (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)

1. Let \( a \) be an arbitrary integer
   2.1 \( \text{Even}(a) \) Assumption
   2.2 \( \exists y \,(a = 2y) \) Definition of Even

   \[ \exists y \,(a^2 = 2y) \] 2.5 \( \exists y \,(a^2 = 2y) \) Definition of Even
   2.6 \( \text{Even}(a^2) \) Direct proof rule

2. \( \text{Even}(a) \rightarrow \text{Even}(a^2) \)

3. \( \forall x \,(\text{Even}(x) \rightarrow \text{Even}(x^2)) \) Intro \( \forall \): 1,2
Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Prove: “The square of every even number is even.”

Formal proof of: ∀x (Even(x) → Even(x^2))

1. Let a be an arbitrary integer
   2.1 Even(a) Assumption
   2.2 ∃y (a = 2y) Definition of Even

   2.5 ∃y (a^2 = 2y) Intro ∃ rule: 2.2
   2.6 Even(a^2) Definition of Even

2. Even(a) → Even(a^2) Direct proof rule

3. ∀x (Even(x) → Even(x^2)) Intro ∀: 1,2
Even and Odd

Prove: “The square of every even number is even.”

Formal proof of: \( \forall x \ (\text{Even}(x) \to \text{Even}(x^2)) \)

1. Let \( a \) be an arbitrary integer
   
   2.1 \quad \text{Even}(a) \quad \text{Assumption}
   
   2.2 \quad \exists y \ (a = 2y) \quad \text{Definition of Even}
   
   2.3 \quad a = 2b \quad \text{Elim \( \exists \): \( b \) special depends on \( a \)}

   \( \to \)

   2.5 \quad \exists y \ (a^2 = 2y) \quad \text{Intro \( \exists \) rule: ?}

   2.6 \quad \text{Even}(a^2) \quad \text{Definition of Even}

2. \quad \text{Even}(a) \to \text{Even}(a^2) \quad \text{Direct proof rule}

3. \quad \forall x \ (\text{Even}(x) \to \text{Even}(x^2)) \quad \text{Intro \( \forall \): 1,2}
**Even and Odd**

```
Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers
```

**Prove:** “The square of every even number is even.”

**Formal proof of:** \( \forall x \ (\text{Even}(x) \rightarrow \text{Even}(x^2)) \)

1. Let \( a \) be an arbitrary integer
   
   \[
   \begin{align*}
   2.1 \quad & \text{Even}(a) & \text{Assumption} \\
   2.2 \quad & \exists y \ (a = 2y) & \text{Definition of Even} \\
   2.3 \quad & a = 2b & \text{Elim } \exists: b \text{ special depends on } a \\
   2.4 \quad & a^2 = 4b^2 = 2(2b^2) & \text{Algebra} \\
   2.5 \quad & \exists y \ (a^2 = 2y) & \text{Intro } \exists \text{ rule} \\
   2.6 \quad & \text{Even}(a^2) & \text{Definition of Even}
   \end{align*}
   \]

2. \( \text{Even}(a) \rightarrow \text{Even}(a^2) \) \hspace{1cm} \text{Direct proof rule}

3. \( \forall x \ (\text{Even}(x) \rightarrow \text{Even}(x^2)) \) \hspace{1cm} \text{Intro } \forall: 1,2
Why did we need to say that \( b \) depends on \( a \)?

There are extra conditions on using these rules:

- \( \forall x \exists y (y \geq x) \) is True but \( \exists y \forall x (y \geq x) \) is False

**BAD “PROOF”**

1. \( \forall x \exists y (y \geq x) \) Given
2. Let \( a \) be an arbitrary integer
3. \( \exists y (y \geq a) \) Elim \( \forall \): 1
4. \( b \geq a \) Elim \( \exists \): \( b \) special depends on \( a \)
5. \( \forall x (b \geq x) \) Intro \( \forall \): 2,4
6. \( \exists y \forall x (y \geq x) \) Intro \( \exists \): 5

* in the domain of \( P \)

** c has to be a NEW name.
Why did we need to say that \( b \) depends on \( a \)?

There are extra conditions on using these rules:

\[
\begin{align*}
\text{Intro } \forall & \quad \text{“Let } a \text{ be arbitrary*”... } P(a) \\
\quad & \quad \therefore \quad \forall x \ P(x) \\
\end{align*}
\]

* in the domain of \( P \)

\[
\begin{align*}
\text{Elim } \exists & \quad \exists x \ P(x) \\
\quad & \quad \therefore \quad P(c) \text{ for some special** } c \\
\end{align*}
\]

** \( c \) has to be a NEW name.

Over integer domain: \( \forall x \ \exists y \ (y \geq x) \) is True but \( \exists y \forall x \ (y \geq x) \) is False

**BAD “PROOF”**

1. \( \forall x \ \exists y \ (y \geq x) \) Given
2. Let \( a \) be an arbitrary integer
3. \( \exists y \ (y \geq a) \) Elim \( \forall \): 1
4. \( b \geq a \) Elim \( \exists \): \( b \) special depends on \( a \)
5. \( \forall x \ (b \geq x) \) Intro \( \forall \): 2, 4
6. \( \exists y \forall x \ (y \geq x) \) Intro \( \exists \): 5

Can’t get rid of \( a \) since another name in the same line, \( b \), depends on it!
Why did we need to say that \( b \) depends on \( a \)?

There are extra conditions on using these rules:

- **Intro \( \forall \)**
  
  “Let \( a \) be arbitrary*”... \( P(a) \)
  
  \[
  \therefore \quad \forall x \ P(x)
  \]

- **Elim \( \exists \)**
  
  \( \exists x \ P(x) \)
  
  \[
  \therefore \quad P(c) \text{ for some special** } c
  \]

* in the domain of \( P \). No other name in \( P \) depends on \( a \)

** \( c \) is a NEW name.
List all dependencies for \( c \).

Over integer domain: \( \forall x \ \exists y \ (y \geq x) \) is True but \( \exists y \forall x \ (y \geq x) \) is False

**BAD “PROOF”**

1. \( \forall x \ \exists y \ (y \geq x) \) \hspace{1cm} \text{Given}
2. Let \( a \) be an arbitrary integer
3. \( \exists y \ (y \geq a) \) \hspace{1cm} \text{Elim } \forall: 1
4. \( b \geq a \) \hspace{1cm} \text{Elim } \exists: b \text{ special depends on } a
5. \( \forall x \ (b \geq x) \) \hspace{1cm} \text{Intro } \forall: 2,4
6. \( \exists y \forall x \ (y \geq x) \) \hspace{1cm} \text{Intro } \exists: 5

Can’t get rid of \( a \) since another name in the same line, \( b \), depends on it!
Inference Rules for Quantifiers: Full version

** Elim \( \exists x \ P(x) \)
\[ \therefore P(c) \text{ for some } \text{special}^{**} c \]

** c is a NEW name.
List all dependencies for c.

** Intro \( \exists \)
\[ \therefore \exists x \ P(x) \]

P(c) for some c

\[ \therefore P(c) \text{ for some } \text{special}^{**} c \]

“Let a be arbitrary*”...P(a)

\[ \therefore \forall x \ P(x) \]

* in the domain of P. No other name in P depends on a

\[ \therefore \forall x \ P(x) \]
English Proofs

• We often write proofs in English rather than as fully formal proofs
  – They are more natural to read

• English proofs follow the structure of the corresponding formal proofs
  – Formal proof methods help to understand how proofs really work in English...
    ... and give clues for how to produce them.
An English Proof

Prove “There is an even integer”

Proof:

2 = 2 \cdot 1  

so 2 equals 2 times an integer.

Therefore 2 is even.

Therefore, there is an even integer

Predicate Definitions

Even(x) \equiv \exists y \ (x = 2 \cdot y)
Odd(x) \equiv \exists y \ (x = 2 \cdot y + 1)

Proof:

1. 2 = 2 \cdot 1  
   Arithmetic

2. \exists y (2 = 2 \cdot y)  
   Intro \exists: 1

3. Even(2)  
   Defn of Even: 2

4. \exists x Even(x)  
   Intro \exists: 3
Prove “The square of every even integer is even.”

Proof: Let \(a\) be an arbitrary even integer.

1. Let \(a\) be an arbitrary integer

Then, by definition, \(a = 2b\) for some integer \(b\) (depending on \(a\)).

Squaring both sides, we get \(a^2 = 4b^2 = 2(2b^2)\).

Since \(2b^2\) is an integer, by definition, \(a^2\) is even.

Since \(a\) was arbitrary, it follows that the square of every even number is even.
Prove “The square of every odd number is odd.”
Prove “The square of every odd number is odd.”

Proof: Let b be an arbitrary odd number. Then, b = 2c+1 for some integer c (depending on b). Therefore, \( b^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1 \). Since \( 2c^2 + 2c \) is an integer, \( b^2 \) is odd. The statement follows since b was arbitrary. ■
Proofs

• Formal proofs follow simple well-defined rules and should be easy to check
  – In the same way that code should be easy to execute

• English proofs correspond to those rules but are designed to be easier for humans to read
  – Easily checkable in principle