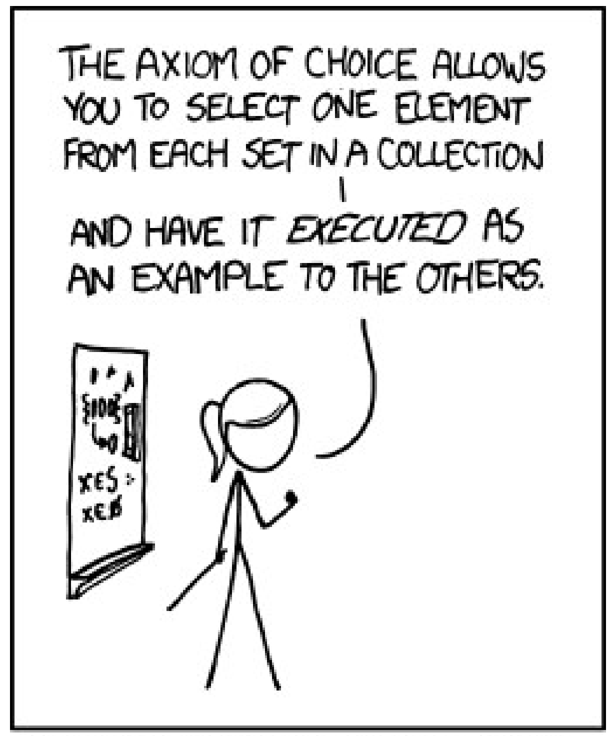
CSE 311: Foundations of Computing

Lecture 8: Predicate Logic Proofs



MY MATH TEACHER WAS A BIG BELIEVER IN PROOF BY INTIMIDATION.

Last class: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \land B}{\therefore A, B}$$

Intro ∧

$$\frac{A \;;\; B}{\therefore A \land B}$$

Elim ∨

$$\frac{A \lor B \;;\; \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \lor B,\; B \lor A}$$

Modus Ponens

$$\frac{A \;;\; A \to B}{\therefore B}$$

Direct Proof Rule

$$\frac{A \Rightarrow B}{\therefore A \to B}$$

Not like other rules

Last class: Example

Prove: $((p \to q) \wedge (q \to r)) \to (p \to r)$

1.1.	$(p \to q) \wedge (q \to r)$	Assumption
1.2.	$p \to q$	$\wedge$ Elim: 1.1
1.3.	$q \to r$	$\wedge$ Elim: 1.1
1.4.1.	p	Assumption
1.4.2.	q	MP: 1.2, 1.4.1
1.4.3.	r	MP: 1.3, 1.4.2
1.4.	$p \to r$	Direct Proof Rule
1.	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Direct Proof Rule

One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.

Inference Rules for Quantifiers: First look

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some } \textit{special}^{**}\ c}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

** By special, we mean that c is a name for a value where P(c) is true. We can't use anything else about that value, so c has to be a NEW name!

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore\ P(a) \text{ for any } a}$$

Intro ∀

$$\frac{\text{“Let } a \text{ be arbitrary*”} \ldots P(a)}{\therefore\ \forall x\, P(x)}$$

* in the domain of P

Predicate Logic Proofs

- Can use
  - Predicate logic inference rules
    whole formulas only
  - Predicate logic equivalences (De Morgan's)
    even on subformulas
  - Propositional logic inference rules
    whole formulas only
  - Propositional logic equivalences
    even on subformulas

My First Predicate Logic Proof

Intro $\exists$
$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim $\forall$
$$\frac{\forall x\, P(x)}{\therefore \quad P(a) \text{ for any } a}$$

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

5. $\forall x\, P(x) \rightarrow \exists x\, P(x)$?

The main connective is implication so Direct Proof Rule seems good

My First Predicate Logic Proof

Intro ∃: $\dfrac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$

Elim ∀: $\dfrac{\forall x\, P(x)}{\therefore\ P(a) \text{ for any } a}$

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

1.5. $\exists x\, P(x)$?

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃ — P(c) for some c ∴ $\exists x\, P(x)$

Elim ∀ — $\forall x\, P(x)$ ∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

We need an ∃ we don't have
so "intro ∃" rule makes sense

1.5. $\exists x\, P(x)$?

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro $\exists$: P(c) for some c ∴ $\exists$x P(x)

Elim $\forall$: $\forall$x P(x) ∴ P(a) for any a

Prove $\forall x\ P(x) \rightarrow \exists x\ P(x)$

1.1. $\forall x\, P(x)$ Assumption

We need an $\exists$ we don't have so "intro $\exists$" rule makes sense

1.5. $\exists x\, P(x)$ Intro $\exists$: ?

That requires P(c) for some c.

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1.	$\forall x\, P(x)$	Assumption
1.2	$P(a)$	Elim ∀: 1.1

We could have picked any name or domain expression here.

1.5.	$\exists x\, P(x)$	Intro ∃: ?
1.	$\forall x\, P(x) \rightarrow \exists x\, P(x)$	Direct Proof Rule

That requires P(c) for some c.

My First Predicate Logic Proof

Intro ∃	P(c) for some c
	∴ $\exists x\, P(x)$

Elim ∀	$\forall x\, P(x)$
	∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

No holes. Just need to clean up.

1.1. $\forall x\, P(x)$ Assumption

1.2 $P(a)$ Elim ∀: 1.1

1.5. $\exists x\, P(x)$ Intro ∃: 1.2

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1.	$\forall x\, P(x)$	Assumption
1.2	$P(a)$	Elim ∀: 1.1
1.3.	$\exists x\, P(x)$	Intro ∃: 1.2
1.	$\forall x\, P(x) \rightarrow \exists x\, P(x)$	Direct Proof Rule

Working forwards as well as backwards:
In applying "Intro ∃" rule we didn't know what expression we might be able to prove P(c) for, so we worked forwards to figure out what might work.

Predicate Logic Proofs with more content

- In propositional logic we could just write down other propositional logic statements as "givens"

- Here, we also want to be able to use domain knowledge so proofs are about something specific

- Example:

Domain of Discourse
Integers

- Given the basic properties of arithmetic on integers, define:

Predicate Definitions
$\text{Even}(x) \equiv \exists y\ (x = 2 \cdot y)$
$\text{Odd}(x) \equiv \exists y\ (x = 2 \cdot y + 1)$

A Not so Odd Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) ≡ ∃y (x = 2·y)
Odd(x) ≡ ∃y (x = 2·y + 1)

Prove "There is an even number"

Formally: prove $\exists x\ \text{Even}(x)$

1. $0 = 2 \cdot 0$ — Arithmetic
2. $\exists y\ (0 = 2y)$ — ∃ Intro: 1
3. Even(0) — Defn of Even
4. $\exists x\ \text{Even}(x)$ — ∃ Intro: 3.

A Not so Odd Example

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) \equiv \exists y\ (x = 2 \cdot y)$
$\text{Odd}(x) \equiv \exists y\ (x = 2 \cdot y + 1)$

Prove "There is an even number"

Formally: prove $\exists x\ \text{Even}(x)$

1.	$\mathbf{2 = 2 \cdot 1}$	**Arithmetic**
2.	$\exists y\ (\mathbf{2} = \mathbf{2} \cdot y)$	**Intro** $\exists$**: 1**
3.	$\text{Even}(\mathbf{2})$	**Definition of** Even**: 2**
4.	$\exists x\ \text{Even}(x)$	**Intro** $\exists$**: 3**

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
$Even(x) \equiv \exists y\ (x = 2 \cdot y)$
$Odd(x) \equiv \exists y\ (x = 2 \cdot y + 1)$
$Prime(x) \equiv$ "$x > 1$ and $x \neq a \cdot b$ for all integers a, b with $1 < a < x$"

Prove "There is an even prime number"

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2 \cdot y)$
Odd(x) ≡ $\exists y\ (x = 2 \cdot y + 1)$
Prime(x) ≡ "x > 1 and x≠a·b for all integers a, b with 1<a<x"

Prove "There is an even prime number"

Formally: prove $\exists x\ (\text{Even}(x) \wedge \text{Prime}(x))$

1.	**$2 = 2 \cdot 1$**	**Arithmetic**
2.	Prime(**2**)*	**Property of integers**
3.	$\exists y\ (2 = 2 \cdot y)$	Intro ∃: 1
4.	Even(2)	Defn of Even: 3
5.	Even(2) ∧ Prime(2)	∧ Intro: 2, 4
6.	$\exists x\ (\text{Even}(x) \wedge \text{Prime}(x))$	Intro ∃: 5

* **Later we will further break down** "Prime" **using quantifiers to prove statements like this**

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) ≡ ∃y (x = 2·y)
Odd(x) ≡ ∃y (x = 2·y + 1)
Prime(x) ≡ "x > 1 and x≠a·b for all integers a, b with 1<a<x"

Prove "There is an even prime number"

Formally: prove ∃x (Even(x) ∧ Prime(x))

1.	**2 = 2·1**	**Arithmetic**
2.	Prime(**2**)*	**Property of integers**
3.	∃y (**2 = 2**·y)	**Intro** ∃**: 1**
4.	Even(**2**)	**Defn of** Even**: 3**
5.	Even(**2**) ∧ Prime(**2**)	**Intro** ∧**: 2, 4**
6.	∃x (Even(x) ∧ Prime(x))	**Intro** ∃**: 5**

*** Later we will further break down** "Prime" **using quantifiers to prove statements like this**

Inference Rules for Quantifiers: First look

Elim ∃

$$\frac{\exists x\ P(x)}{\therefore\ P(c) \text{ for some } \textit{special}^{**}\ c}$$

** By special, we mean that c is a name for a value where P(c) is true. We can't use anything else about that value, so c has to be a NEW name!

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\ P(x)}$$

Elim ∀

$$\frac{\forall x\ P(x)}{\therefore\ P(a) \text{ for any } a}$$

Intro ∀

$$\frac{\text{"Let } a \text{ be arbitrary*"} \ldots P(a)}{\therefore\ \forall x\ P(x)}$$

* in the domain of P

Even(x) ≡ $\exists y\ (x=2y)$
Odd(x) ≡ $\exists y\ (x=2y+1)$
Domain: Integers

Even and Odd

Intro ∀	"Let a be arbitrary*"...P(a) ∴ $\forall x\ P(x)$	Elim ∃	$\exists x\ P(x)$ ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$

3. $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$?

Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Intro ∀	"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	Elim ∃	∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: ∀x (Even(x) → Even(x^2))

1. Let **a** be an arbitrary integer

2. Even(**a**)→Even(**a**^{2}) ?
3. ∀x (Even(x)→Even(x^2)) Intro ∀: 1,2

Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Intro ∀	Elim ∃
"Let a be arbitrary*"...P(a) / ∴ $\forall x\, P(x)$	$\exists x\, P(x)$ / ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: $\forall x\, (\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

    2.1 Even(**a**) — Assumption

    2.6 Even($\mathbf{a}^2$) — ?

2. Even(**a**)→Even($\mathbf{a}^2$) — Direct proof rule
3. $\forall x\, (\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro ∀: 1,2

Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Intro ∀	Elim ∃
"Let a be arbitrary*"...P(a) / ∴ $\forall x\ P(x)$	$\exists x\ P(x)$ / ∴ P(c) **for some *special***** c

Prove: "The square of every even number is even."

Formal proof of: $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. **Let a be an arbitrary integer**
   - 2.1 Even(**a**) — Assumption
   - 2.2 $\exists y\ (\mathbf{a} = 2y)$ — Definition of Even
   - 2.5 $\exists y\ (\mathbf{a}^2 = 2y)$ — ?
   - 2.6 Even($\mathbf{a}^2$) — Definition of Even
2. Even(**a**)$\rightarrow$Even($\mathbf{a}^2$) — Direct proof rule
3. $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro ∀: 1,2

Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Intro ∀: "Let a be arbitrary*"...P(a) ∴ ∀x P(x)

Elim ∃: ∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer
   - 2.1 Even(**a**) — Assumption
   - 2.2 ∃y (**a** = 2y) — Definition of Even
   - 2.5 $\exists y\,(\mathbf{a}^2 = 2y)$ — Intro ∃ rule: ?
   - 2.6 $\text{Even}(\mathbf{a}^2)$ — Definition of Even
2. $\text{Even}(\mathbf{a}) \rightarrow \text{Even}(\mathbf{a}^2)$ — Direct proof rule
3. $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro ∀: 1,2

Need $\mathbf{a}^2 = 2c$ for some **c**

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Even and Odd

Intro ∀	"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	Elim ∃	∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2.1	Even(**a**)	Assumption
2.2	∃y (**a** = 2y)	Definition of Even
2.3	**a** = 2**b**	Elim ∃: **b** special depends on **a**
2.5	∃y ($\mathbf{a}^2$ = 2y)	Intro ∃ rule: ?
2.6	Even($\mathbf{a}^2$)	Definition of Even

Need $\mathbf{a}^2 = 2c$ for some **c**

2.	Even(**a**)→Even($\mathbf{a}^2$)	Direct proof rule
3.	∀x (Even(x)→Even(x^2))	Intro ∀: 1,2

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Even and Odd

Intro ∀	"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	Elim ∃	∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: ∀x (Even(x) → Even(x^2))

1. Let **a** be an arbitrary integer
   - 2.1 Even(**a**) — Assumption
   - 2.2 ∃y (**a** = 2y) — Definition of Even
   - 2.3 **a** = 2**b** — Elim ∃: **b** special depends on **a**
   - 2.4 $\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$ — Algebra
   - 2.5 ∃y ($\mathbf{a}^2$ = 2y) — Intro ∃ rule
   - 2.6 Even($\mathbf{a}^2$) — Definition of Even
2. Even(**a**)→Even($\mathbf{a}^2$) — Direct proof rule
3. ∀x (Even(x)→Even(x^2)) — Intro ∀: 1,2

Used $\mathbf{a}^2 = 2c$ for $c=2\mathbf{b}^2$

y =

Why did we need to say that b depends on a?

There are extra conditions on using these rules:

Intro ∀: "Let a be arbitrary*"...P(a) ∴ ∀x P(x)

* in the domain of P

Elim ∃: ∃x P(x) ∴ P(c) for some *special*** c

** c has to be a NEW name.

Over integer domain: ∀x ∃y (y ≥ x) is True but ∃y∀x (y ≥ x) is False

BAD "PROOF"

1.	∀x ∃y (y ≥ x)	Given
2.	**Let a be an arbitrary integer**	
3.	∃y (y ≥ **a**)	Elim ∀: **1**
4.	**b ≥ a**	Elim ∃: **b** special depends on **a**
5.	∀x (**b** ≥ x)	Intro ∀: **2,4**
6.	∃y∀x (y ≥ x)	Intro ∃ : **5**

Why did we need to say that b depends on a?

There are extra conditions on using these rules:

Intro ∀: "Let a be arbitrary*"…P(a) ∴ ∀x P(x)

* in the domain of P

Elim ∃: ∃x P(x) ∴ P(c) for some *special*** c

** c has to be a NEW name.

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

BAD "PROOF"

1. $\forall x\, \exists y\, (y \geq x)$ — Given
2. **Let a be an arbitrary integer**
3. $\exists y\, (y \geq a)$ — Elim ∀: **1**
4. $b \geq a$ — Elim ∃: **b** special depends on **a**
5. $\forall x\, (b \geq x)$ — Intro ∀: **2,4**
6. $\exists y \forall x\, (y \geq x)$ — Intro ∃ : **5**

Can't get rid of a since another name in the same line, b, depends on it!

Why did we need to say that b depends on a?

There are extra conditions on using these rules:

Intro ∀: “Let a be arbitrary*”…P(a) ∴ ∀x P(x)

* in the domain of P. No other name in P depends on a

Elim ∃: ∃x P(x) ∴ P(c) for some *special*** c

** c is a NEW name. List all dependencies for c.

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

BAD “PROOF”

1. $\forall x\, \exists y\, (y \geq x)$ — Given
2. **Let a be an arbitrary integer**
3. $\exists y\, (y \geq a)$ — Elim ∀: 1
4. $b \geq a$ — Elim ∃: b special depends on a
5. ~~$\forall x\, (b \geq x)$ — Intro ∀: 2,4~~
6. $\exists y \forall x\, (y \geq x)$ — Intro ∃ : 5

Can’t get rid of a since another name in the same line, b, depends on it!

Inference Rules for Quantifiers: Full version

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some special** } c}$$

** c is a NEW name.
List all dependencies for c.

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∀

$$\frac{\text{“Let } a \text{ be arbitrary*”} \ldots P(a)}{\therefore \forall x\, P(x)}$$

* in the domain of P. No other name in P depends on a

English Proofs

- We often write proofs in English rather than as fully formal proofs
  - They are more natural to read

- English proofs follow the structure of the corresponding formal proofs
  - Formal proof methods help to understand how proofs really work in English…

    … and give clues for how to produce them.

An English Proof

Predicate Definitions

Even(x) ≡ ∃y (x = 2·y)

Odd(x) ≡ ∃y (x = 2·y + 1)

Prove "There is an even integer"

Proof:

2 = 2·1	1.	2 = 2·1	Arithmetic
so 2 equals 2 times an integer.	2.	∃y (2 = 2·y)	Intro ∃: 1
Therefore 2 is even.	3.	Even(2)	Defn of Even: 2
Therefore, there is an even integer ■	4.	∃x Even(x)	Intro ∃: 3

English Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

English proof	Formal proof
Proof: Let a be an arbitrary even integer.	1. Let **a** be an arbitrary integer 2.1 Even(**a**) — Assumption
Then, by definition, a = 2b for some integer b (depending on a).	2.2 $\exists y\ (a = 2y)$ — Definition 2.3 $a = 2b$ — **b** special depends on **a**
Squaring both sides, we get $a^2 = 4b^2 = 2(2b^2)$.	2.4 $a^2 = 4b^2 = 2(2b^2)$ — Algebra
Since $2b^2$ is an integer, by definition, a^2 is even.	2.5 $\exists y\ (a^2 = 2y)$ 2.6 Even(a^2) — Definition
Since a was arbitrary, it follows that the square of every even number is even. ■	2. Even(a)$\rightarrow$Even(a^2) 3. $\forall x\ (\text{Even}(x)\rightarrow\text{Even}(x^2))$

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The square of every odd number is odd."

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The square of every odd number is odd."

Proof: Let b be an arbitrary odd number.

Then, b = 2c+1 for some integer c (depending on b).

Therefore, $b^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1$.

Since $2c^2+2c$ is an integer, b^2 is odd. The statement follows since b was arbitrary. ■

Proofs

- Formal proofs follow simple well-defined rules and should be easy to check
  - In the same way that code should be easy to execute
- English proofs correspond to those rules but are designed to be easier for humans to read
  - Easily checkable in principle