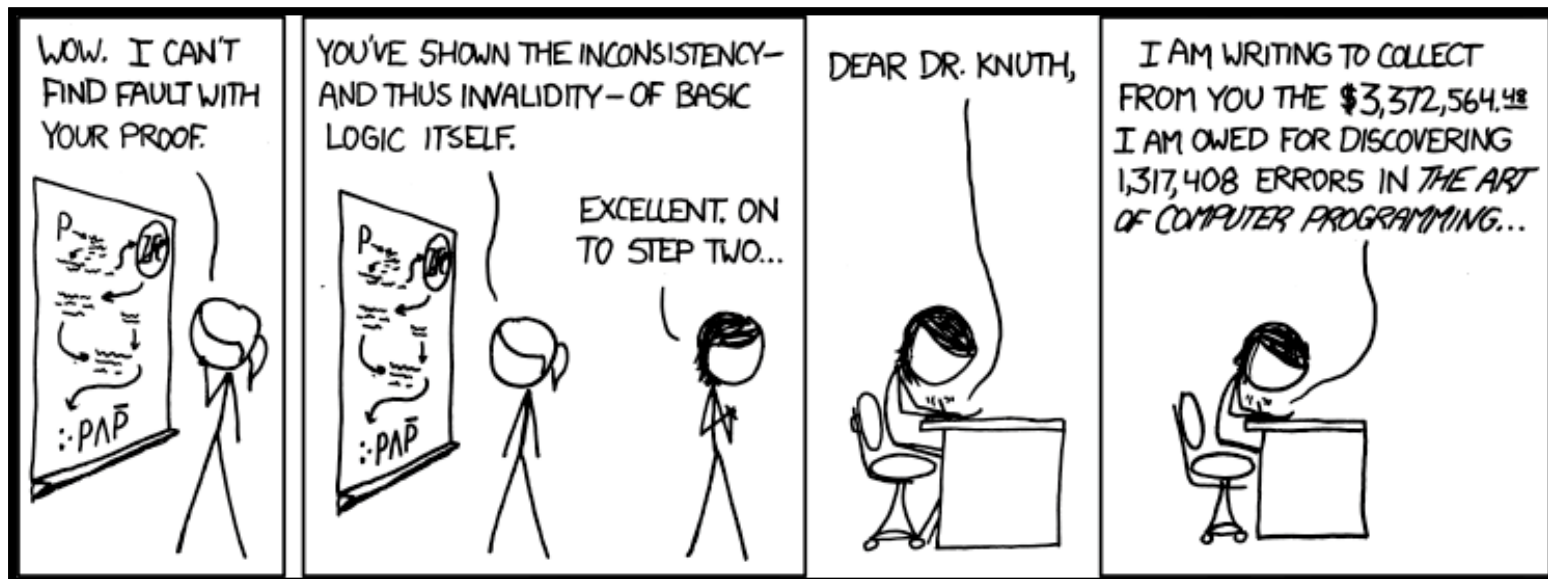


CSE 311: Foundations of Computing

Lecture 7: Logical Inference



Logical Inference

- So far we've considered:
 - How to understand and *express* things using propositional and predicate logic
 - How to *compute* using Boolean (propositional) logic
 - How to show that different ways of expressing or computing them are *equivalent* to each other
- Logic also has methods that let us *infer* implied properties from ones that we know
 - Equivalence is a small part of this

Applications of Logical Inference

- **Software Engineering**
 - Express desired properties of program as set of logical constraints
 - Use inference rules to show that program implies that those constraints are satisfied
- **Artificial Intelligence**
 - Automated reasoning
- **Algorithm design and analysis**
 - e.g., Correctness, Loop invariants.
- **Logic Programming, e.g. Prolog**
 - Express desired outcome as set of constraints
 - Automatically apply logic inference to derive solution

Proofs

- **Start with hypotheses and facts**
- **Use rules of inference to extend set of facts**
- **Result is proved when it is included in the set**

An inference rule: *Modus Ponens*

- If **A** and **A** \rightarrow **B** are both true then **B** must be true
- Write this rule as
$$\frac{A ; A \rightarrow B}{\therefore B}$$
- Given:
 - If it is Wednesday then you have a 311 class today.
 - It is Wednesday.
- Therefore, by Modus Ponens:
 - You have a 311 class today.

My First Proof!

Show that r follows from p , $p \rightarrow q$, and $q \rightarrow r$

1. p Given
2. $p \rightarrow q$ Given
3. $q \rightarrow r$ Given
- 4.
- 5.

Modus Ponens $\frac{A; A \rightarrow B}{\therefore B}$

My First Proof!

Show that r follows from p , $p \rightarrow q$, and $q \rightarrow r$

1. p Given
2. $p \rightarrow q$ Given
3. $q \rightarrow r$ Given
4. q MP: 1, 2
5. r MP: 3, 4

Modus Ponens $\frac{A; A \rightarrow B}{\therefore B}$

Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

- | | | |
|----|-----------------------------|-------------------|
| 1. | $p \rightarrow q$ | Given |
| 2. | $\neg q$ | Given |
| 3. | $\neg q \rightarrow \neg p$ | Contrapositive: 1 |
| 4. | $\neg p$ | MP: 2, 3 |

Modus Ponens $\frac{A; A \rightarrow B}{\therefore B}$

Inference Rules

If **A** is true and **B** is true

Requirements: **A ; B**

Conclusions: **∴ C , D**

Then, **C** must
be true

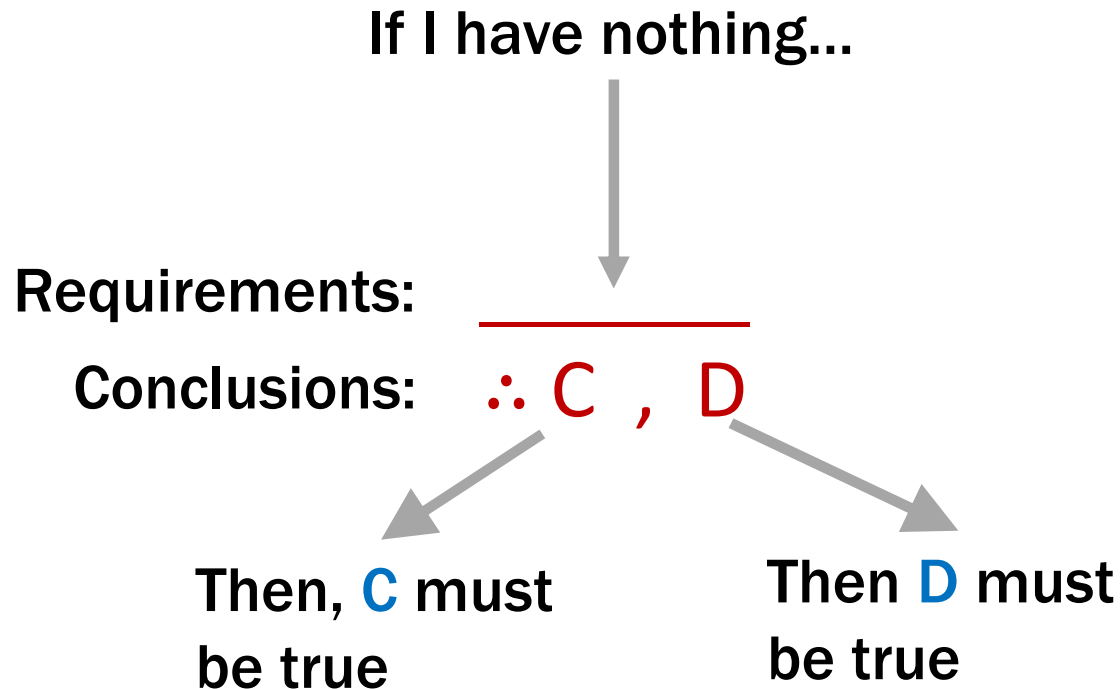
Then **D** must
be true

Example (Modus Ponens):

A ; A → B
∴ B

If I have **A** and **A → B** both true,
Then **B** must be true.

Axioms: Special inference rules



Example (Excluded Middle):

$\therefore A \vee \neg A$

$A \vee \neg A$ must be true.

Simple Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

$$\boxed{\text{Elim } \wedge} \frac{A \wedge B}{\therefore A, B}$$

$$\boxed{\text{Intro } \wedge} \frac{A ; B}{\therefore A \wedge B}$$

$$\boxed{\text{Elim } \vee} \frac{A \vee B ; \neg A}{\therefore B}$$

$$\boxed{\text{Intro } \vee} \frac{A}{\therefore A \vee B, B \vee A}$$

$$\boxed{\text{Modus Ponens}} \frac{A ; A \rightarrow B}{\therefore B}$$

$$\boxed{\text{Direct Proof Rule}} \frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules

Proofs

Show that r follows from p , $p \rightarrow q$ and $(p \wedge q) \rightarrow r$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{A ; A \rightarrow B}{\therefore B}$$

$$\frac{A \wedge B}{\therefore A, B}$$

$$\frac{A ; B}{\therefore A \wedge B}$$

Proofs

Show that r follows from $p, p \rightarrow q$, and $p \wedge q \rightarrow r$

Two visuals of the same proof.
We will use the top one, but if
the bottom one helps you
think about it, that's great!

- | | | |
|----|----------------------------|-----------------------|
| 1. | p | Given |
| 2. | $p \rightarrow q$ | Given |
| 3. | q | MP: 1, 2 |
| 4. | $p \wedge q$ | Intro \wedge : 1, 3 |
| 5. | $p \wedge q \rightarrow r$ | Given |
| 6. | r | MP: 4, 5 |

$$\frac{\frac{p \ ; \ p \rightarrow q}{p \ ; \ q} \text{MP}}{p \wedge q \ ; \ p \wedge q \rightarrow r} \text{Intro } \wedge$$
$$\frac{p \wedge q \ ; \ p \wedge q \rightarrow r}{r} \text{MP}$$

Important: Applications of Inference Rules

- You can use **equivalences** to make substitutions of **any sub-formula**.
- **Inference rules only** can be applied to **whole formulas** (not correct otherwise).

e.g. 1. $p \rightarrow r$ given

~~2. $(p \vee q) \rightarrow r$ intro \vee from 1.~~

Does not follow! e.g. $p=F, q=T, r=F$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

First: Write down givens
and goal

20. $\neg r$



Idea: Work backwards!

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given

2. $q \rightarrow \neg r$ Given

3. $\neg s \vee q$ Given

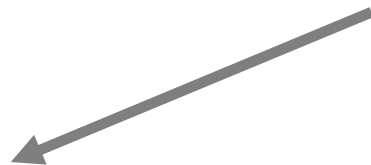
Idea: Work backwards!

We want to eventually get $\neg r$. How?

- We can use $q \rightarrow \neg r$ to get there.
- The justification between 2 and 20 looks like “elim \rightarrow ” which is MP.

20. $\neg r$

MP: 2,



Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- Now, we have a new “hole”
- We need to prove q ...
 - Notice that at this point, if we prove q , we’ve proven $\neg r$...

19. q



20. $\neg r$

MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given

2. $q \rightarrow \neg r$ Given

3. $\neg s \vee q$ Given

This looks like or-elimination.

19. q

?

20. $\neg r$


MP: 2, 19

Elim \vee $\frac{A \vee B; \neg A}{\therefore B}$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.


1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

18. $\neg\neg s$  $\neg\neg s$ doesn't show up in the givens but s does and we can use equivalences
19. q \vee Elim: 3, 18
20. $\neg r$ MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

17. s 
18. $\neg\neg s$ Double Negation: 17
19. q \vee Elim: 3, 18
20. $\neg r$ MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given

2. $q \rightarrow \neg r$ Given

3. $\neg s \vee q$ Given

No holes left! We just need to clean up a bit.

17. s \wedge Elim: 1

18. $\neg\neg s$ Double Negation: 17

19. q \vee Elim: 3, 18

20. $\neg r$ MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given
4. s \wedge Elim: 1
5. $\neg\neg s$ Double Negation: 4
6. q \vee Elim: 3, 5
7. $\neg r$ MP: 2, 6

To Prove An Implication: $A \rightarrow B$

- We use the direct proof rule
- The “pre-requisite” $A \Rightarrow B$ for the direct proof rule is a proof that “Given A , we can prove B .”

- **The direct proof rule:**

If you have such a proof then you can conclude that $A \rightarrow B$ is true

Example: Prove $p \rightarrow (p \vee q)$.

Indent proof
subroutine \Rightarrow

proof subroutine

- | | |
|-------------------------------|-------------------|
| 1. p | Assumption |
| 2. $p \vee q$ | Intro \vee : 1 |
| 3. $p \rightarrow (p \vee q)$ | Direct Proof Rule |

Proofs using the direct proof rule

Show that $p \rightarrow r$ follows from q and $(p \wedge q) \rightarrow r$

1. q Given

2. $(p \wedge q) \rightarrow r$ Given

This is a
proof
of $p \rightarrow r$

3.1. p Assumption

3.2. $p \wedge q$ Intro \wedge : 1, 3.1

3.3. r MP: 2, 3.2

If we know p is true...
Then, we've shown
 r is true

3. $p \rightarrow r$ Direct Proof Rule

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens...

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

1.1. $p \wedge q$

1.2. p

1.3. $p \vee q$

1. $(p \wedge q) \rightarrow (p \vee q)$

Assumption

Elim \wedge : 1.1

Intro \vee : 1.2

Direct Proof Rule

Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

1.1. $(p \rightarrow q) \wedge (q \rightarrow r)$ Assumption

1.2. $p \rightarrow q$ \wedge Elim: 1.1

1.3. $q \rightarrow r$ \wedge Elim: 1.1

1.4.1. p Assumption

1.4.2. q MP: 1.2, 1.4.1

1.4.3. r MP: 1.3, 1.4.2

1.4. $p \rightarrow r$ Direct Proof Rule

1. $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$ Direct Proof Rule

One General Proof Strategy

- 1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given**
- 2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.**
- 3. Write the proof beginning with what you figured out for 2 followed by 1.**