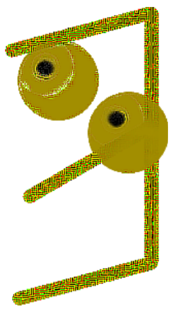
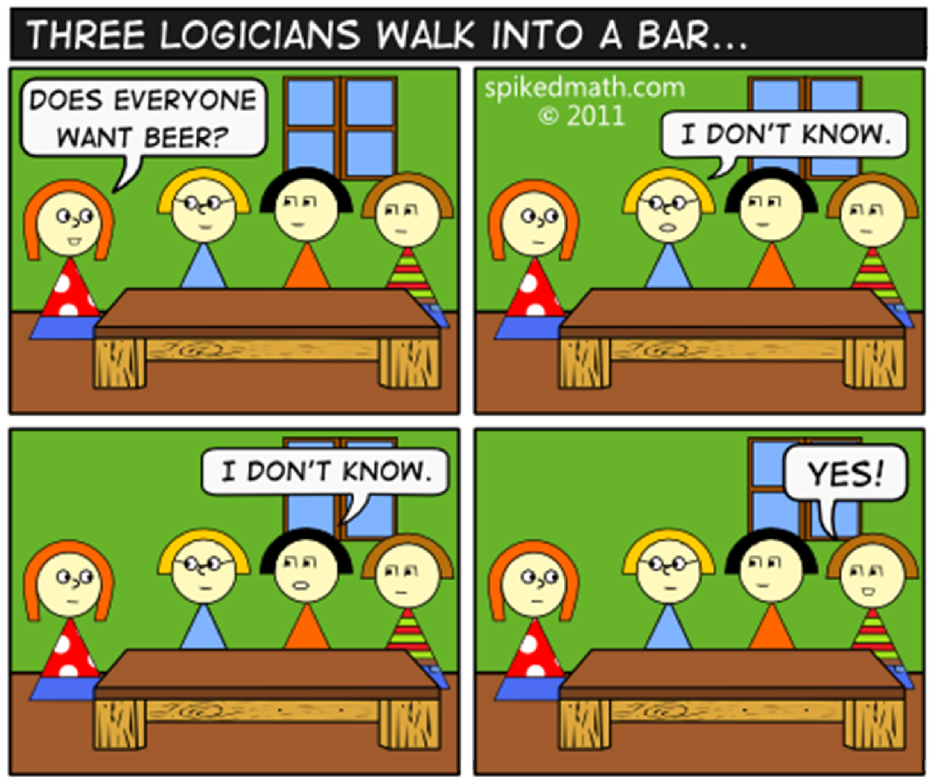
CSE 311: Foundations of Computing

Lecture 6: More Predicate Logic





Another notation for negation in Boolean Algebra
$\bar{a}$ instead of a'
Doesn't work well typeset or in Powerpoint.
$\neg$, $'$, $\bar{\ }$, -D>- $\models$

Don't mix propositional logic & Boolean Algebra

Last class: Predicates

Predicate

– A function that returns a truth value, e.g.,

Cat(x) ::= "x is a cat"

Prime(x) ::= "x is prime"

HasTaken(x, y) ::= "student x has taken course y"

LessThan(x, y) ::= "$x < y$"

Sum(x, y, z) ::= "$x + y = z$"

GreaterThan5(x) ::= "$x > 5$"

HasNChars(s, n) ::= "string s has length n"

Predicates can have varying numbers of arguments and input types.

Last class: Domain of Discourse

For ease of use, we define one "type"/"domain" that we work over. This set of objects is called the "domain of discourse".

For each of the following, what might the domain be?

(1) "x is a cat", "x barks", "x ruined my couch"

(2) "x is prime", "$x = 0$", "$x < 0$", "x is a power of two"

(3) "x is a pre-req for z"

Domain of Discourse

For ease of use, we define one "type"/"domain" that we work over. This set of objects is called the "**domain of discourse**".

For each of the following, what might the domain be?

(1) "x is a cat", "x barks", "x ruined my couch"

"mammals" or "sentient beings" or "cats and dogs" or …

(2) "x is prime", "$x = 0$", "$x < 0$", "x is a power of two"

"numbers" or "integers" or "integers greater than 5" or …

(3) "x is a pre-req for z"

"courses"

Last Class: Quantifiers

We use *quantifiers* to talk about collections of objects.

$\forall x\, P(x)$

P(x) **is true for every** x **in the domain**

read as "for all x, P of x"

for ~~any~~

$\exists x\, P(x)$

There is an x **in the domain for which** P(x) **is true**

read as "there exists x, P of x"

Last class: Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Determine the truth values of each of these statements:

$\exists x$ Even(x)

$\forall x$ Odd(x)

$\forall x$ (Even(x) $\vee$ Odd(x))

$\exists x$ (Even(x) $\wedge$ Odd(x))

$\forall x$ Greater(x+1, x)

$\exists x$ (Even(x) $\wedge$ Prime(x))

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= “x is even”	Greater(x, y) ::= “x > y”
Odd(x) ::= “x is odd”	Equal(x, y) ::= “x = y”
Prime(x) ::= “x is prime”	Sum(x, y, z) ::= “x + y = z”

Determine the truth values of each of these statements:

$\exists x\ \mathrm{Even}(x)$	**T**	**e.g. 2, 4, 6, ...**
$\forall x\ \mathrm{Odd}(x)$	**F**	**e.g. 2, 4, 6, ...**
$\forall x\ (\mathrm{Even}(x) \vee \mathrm{Odd}(x))$	**T**	**every integer is either even or odd**
$\exists x\ (\mathrm{Even}(x) \wedge \mathrm{Odd}(x))$	**F**	**no integer is both even and odd**
$\forall x\ \mathrm{Greater}(x+1, x)$	**T**	**adding 1 makes a bigger number**
$\exists x\ (\mathrm{Even}(x) \wedge \mathrm{Prime}(x))$	**T**	**Even(2) is true and Prime(2) is true**

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\ \exists y\ \text{Greater}(y, x)$

$\forall x\ \exists y\ \text{Greater}(x, y)$

$\forall x\ \exists y\ (\text{Greater}(y, x) \wedge \text{Prime}(y))$

$\forall x\ (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

$\exists x\ \exists y\ (\text{Sum}(x, 2, y) \wedge \text{Prime}(x) \wedge \text{Prime}(y))$

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

T $\forall x\, \exists y$ Greater(y, x) There is no largest positive integer

For every positive integer x, there is a positive integer y, such that y > x.

F $\forall x\, \exists y$ Greater(x, y) There is no smallest positive integer

For every positive integer x, there is a positive integer y, such that x > y.

T $\forall x\, \exists y$ (Greater(y, x) $\wedge$ Prime(y)) There is no largest prime number

For every positive integer x, there is a pos. int. y such that y > x and y is prime.

T $\forall x$ (Prime(x) $\rightarrow$ (Equal(x, 2) $\vee$ Odd(x))) Every prime number is either 2 or odd

For each positive integer x, if x is prime, then x = 2 or x is odd.

T $\exists x\, \exists y$ (Sum(x, 2, y) $\wedge$ Prime(x) $\wedge$ Prime(y)) There are prime numbers that differ by 2

There exist positive integers x and y such that x + 2 = y and x and y are prime.

Statements with Quantifiers (Natural Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\, \exists y\, \text{Greater}(y, x)$

There is no greatest integer.

$\forall x\, \exists y\, \text{Greater}(x, y)$

There is no least integer.

$\forall x\, \exists y\, (\text{Greater}(y, x) \wedge \text{Prime}(y))$

For every positive integer there is a larger number that is prime.

$\forall x\, (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

Every prime number is either 2 or odd.

$\exists x\, \exists y\, (\text{Sum}(x, 2, y) \wedge \text{Prime}(x) \wedge \text{Prime}(y))$

There exist prime numbers that differ by two."

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

if q is true $p \to q$ is true

"Red cats like tofu"

$\cancel{\exists x\,((Cat(x) \land Red(x)) \to LikesTofu(x))}$

$\forall x\,((Cat(x) \land Red(x)) \to LikesTofu(x))$

$\cancel{\forall x\,(Cat(x) \land Red(x) \land LikesTofu(x))}$

dogs are mammals

"Some red cats don't like tofu"

$\exists x\,(Cat(x) \land Red(x) \land \neg LikesTofu(x))$

$\cancel{\exists x\,((Cat(x) \land Red(x)) \to \neg LikesTofu(x))}$

some dog does not like tofu

"not all red cats like tofu"

$\neg(\forall x\,((Cat(x) \land Red(x)) \to LikesTofu(x)))$

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

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"Red cats like tofu"

$$\forall x\ ((\text{Red}(x) \wedge \text{Cat}(x)) \rightarrow \text{LikesTofu}(x))$$

"Some red cats don't like tofu"

$$\exists y\ ((\text{Red}(y) \wedge \text{Cat}(y)) \wedge \neg\text{LikesTofu}(y))$$

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

When putting two predicates together like this, we use an "and".

"Red cats like tofu"

When restricting to a smaller domain in a "for all" we use implication.

When there's no leading quantification, it means "for all".

"Some red cats don't like tofu"

When restricting to a smaller domain in an "exists" we use and.

"Some" means "there exists".

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= "x is a purple fruit"

(*) $\forall x$ PurpleFruit(x) ("All fruits are purple")

What is the negation of (*)?

(a) "there exists a purple fruit"

(b) "there exists a non-purple fruit"

(c) "all fruits are not purple"

Try your intuition! Which one "feels" right?

Key Idea: In every domain, exactly one of a statement and its negation should be true.

Negations of Quantifiers

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Key Idea: In every domain, exactly one of a statement and its negation should be true.

Domain of Discourse
{plum}

(*), (a)

Domain of Discourse
{apple}

(b), (c)

Domain of Discourse
{plum, apple}

(a), (b)

Negations of Quantifiers

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(*) $\forall x$ PurpleFruit(x) (“All fruits are purple”)

What is the negation of (*)?

(a) “there exists a purple fruit”
(b) “there exists a non-purple fruit”
(c) “all fruits are not purple”

Key Idea: In every domain, exactly one of a statement and its negation should be true.

Domain of Discourse
{plum}

(*), (a)

Domain of Discourse
{apple}

(b), (c)

Domain of Discourse
{plum, apple}

(a), (b)

The only choice that ensures exactly one of the statement and its negation is (b).

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

Domain is pos integer

$\neg \forall x\, P(x)$ $\qquad \neg(P(1) \wedge P(2) \wedge P(3) \wedge \ldots)$

$\equiv \exists x \neg P(x)$ $\qquad \neg P(1) \vee \neg P(2) \ldots$

$\neg \exists x\, P(x)$ $\qquad \neg(P(1) \vee P(2) \vee \ldots)$

$\equiv \forall x \neg P(x)$ $\qquad \neg P(1) \wedge \neg P(2) \wedge \ldots$

$\neg$

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

"There is no largest integer"

$$\neg \exists x\, \forall y\ (x \geq y)$$
$$\equiv\ \forall x\, \neg \forall y\ (x \geq y)$$
$$\equiv\ \forall x\ \exists y\, \neg (x \geq y)$$
$$\equiv\ \forall x\ \exists y\ (y > x)$$

"For every integer there is a larger integer"

$$\neg \forall x\, ((Red(x) \wedge Cat(x)) \rightarrow LikesTofu(x))$$
$$\equiv \exists x\, \neg(\ -\ \ -\ \ -\ )$$
$$\equiv \exists x\, \neg(\neg(Red(x) \wedge Cat(x)) \vee LikesTofu(x))$$
$$\equiv \exists x\, (\neg\neg(Red(x) \wedge Cat(x)) \wedge \neg LikesTofu(x))$$
$$\equiv \exists x\, (Red(x) \wedge Cat(x) \wedge \neg(LikesTofu(x)))$$

Scope of Quantifiers

$\exists x\ (P(x) \land Q(x))$ **vs.** $\exists x\ P(x) \land \exists x\ Q(x)$

eg. P x is odd
Q x is even

$\exists x\, P(x) \land \exists y\, Q(y)$

F

T

scope of quantifiers

$\exists x \; (P(x) \wedge Q(x))$ **vs.** $\exists x \; P(x) \wedge \exists x \; Q(x)$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

Scope of Quantifiers

Example: NotLargest(x) $\equiv \exists\, y$ Greater (y, x)
$\equiv \exists\, z$ Greater (z, x)

truth value:

doesn't depend on y or z "bound variables"

does depend on x "free variable"

quantifiers only act on free variables of the formula they quantify

$$\underbrace{\forall x\, (\exists y\, \underbrace{(P(x,y) \rightarrow \overbrace{\forall x\, Q(y, x)}^{x})}_{y})}_{x}$$

Quantifier “Style”

$$\forall x(\exists y\,(P(x,y) \rightarrow \forall\, \overset{z}{x}\, Q(y, \overset{z}{x})))$$

This isn’t “wrong”, it’s just horrible style.
Don’t confuse your reader by using the same variable multiple times…there are a lot of letters…

Nested Quantifiers

- **Bound variable names don't matter**

$$\forall x\ \exists y\ P(x, y) \equiv \forall a\ \exists b\ P(a, b)$$

- **Positions of quantifiers can sometimes change**

$$\forall x\ (Q(x) \land \exists y\ P(x, y)) \equiv \forall x\ \exists y\ (Q(x) \land P(x, y))$$

- **But: order is important...**

Quantifier Order Can Matter

Domain of Discourse
Integers OR $\{1, 2, 3, 4\}$

Predicate Definitions
GreaterEq(x, y) ::= “$x \geq y$”

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

“There is a number greater than or equal to all numbers.”

$$\exists x\ \forall y\ \mathrm{GreaterEq}(x, y)))$$

“Every number has a number greater than or equal to it.”

$$\forall y\ \exists x\ \mathrm{GreaterEq}(x, y)))$$

The purple statement requires an entire row to be true.
The red statement requires one entry in each column to be true.

Quantification with Two Variables

expression	when **true**	when **false**
$\forall x \, \forall y \, P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \, \exists y \, P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \, \exists y \, P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \, \forall x \, P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other
- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If p and p $\rightarrow$ q are both true then q must be true
- Write this rule as $\dfrac{p,\ p \rightarrow q}{\therefore\ q}$
- Given:
  - If it is Monday then you have a 311 class today.
  - It is Monday.
- Therefore, by Modus Ponens:
  - You have a 311 class today.

My First Proof!

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.		
5.		

My First Proof!

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.	q	MP: 1, 2
5.	r	MP: 3, 4

Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	Given
2.	$\neg q$	Given
3.	$\neg q \rightarrow \neg p$	Contrapositive: 1
4.	$\neg p$	MP: 2, 3

Inference Rules

- Each inference rule is written as:
  ...which means that if both A and B are true then you can infer C and you can infer D.

$$\frac{A,\ B}{\therefore\ C, D}$$

  – For rule to be correct $(A \wedge B) \rightarrow C$ and $(A \wedge B) \rightarrow D$ must be a tautologies

- Sometimes rules don't need anything to start with. These rules are called axioms:
  – e.g. *Excluded Middle Axiom*

$$\frac{\quad}{\therefore\ p \vee \neg p}$$

Simple Propositional Inference Rules

Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

$$\frac{p \wedge q}{\therefore\ p,\ q} \qquad \frac{p,\ q}{\therefore\ p \wedge q}$$

$$\frac{p \vee q\ ,\ \neg p}{\therefore\ q} \qquad \frac{p}{\therefore\ p \vee q,\ q \vee p}$$

$$\frac{p,\ p \rightarrow q}{\therefore\ q} \qquad \frac{p \Rightarrow q}{\therefore\ p \rightarrow q}$$

Direct Proof Rule

Not like other rules

Proofs

Show that r follows from p, $p \rightarrow q$ and $(p \wedge q) \rightarrow r$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

$$\frac{p \wedge q}{\therefore\ p,\ q}$$

$$\frac{p,\ q}{\therefore\ p \wedge q}$$

Proofs

Show that r follows from $p, p \to q$, and $p \wedge q \to r$

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!

1.	p	Given
2.	$p \to q$	Given
3.	q	MP: 1, 2
4.	$p \wedge q$	Intro $\wedge$: 1, 3
5.	$p \wedge q \to r$	Given
6.	r	MP: 4, 5

$$\dfrac{\dfrac{p \qquad \dfrac{p \qquad p \to q}{q}\,\text{MP}}{p \wedge q}\,\text{Intro }\wedge \qquad p \wedge q \to r}{r}\,\text{MP}$$

Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \rightarrow q$ given

~~2. $(p \vee r) \rightarrow q$ intro $\vee$ from 1.~~

Does not follow! e.g . p=F, q=F, r=T