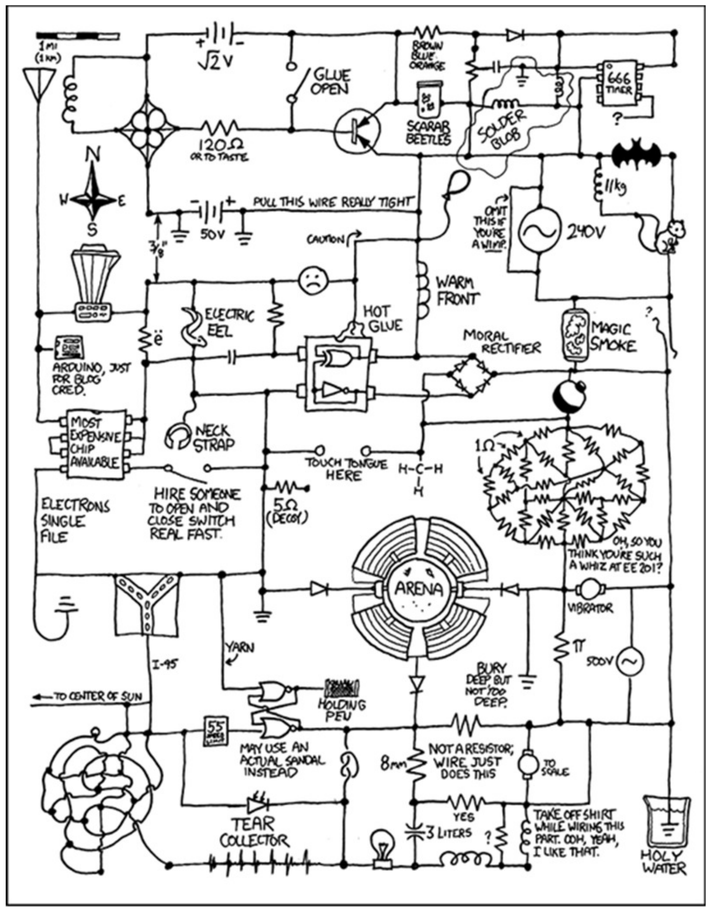
CSE 311: Foundations of Computing

Lecture 5: DNF, CNF and Predicate Logic

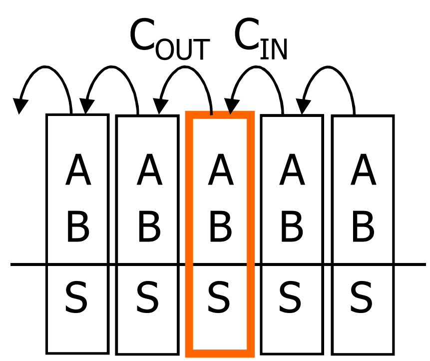


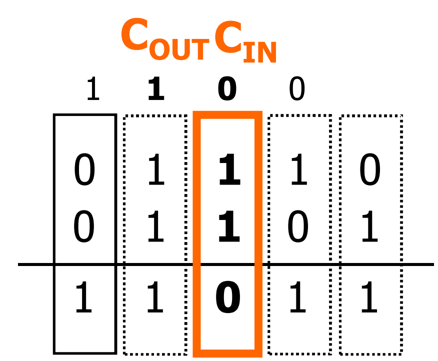
1-bit Binary Adder

$\begin{array}{r} A \\ \underline{+\ B} \\ S \\ (C_{OUT}) \end{array}$	$0 + 0 = 0$ (with $C_{OUT} = 0$) $0 + 1 = 1$ (with $C_{OUT} = 0$) $1 + 0 = 1$ (with $C_{OUT} = 0$) $1 + 1 = 0$ (with $C_{OUT} = 1$)

Idea: These are chained together, with a carry-in

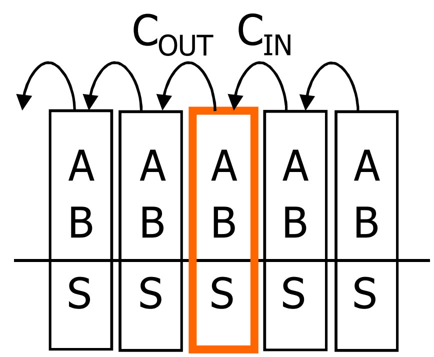
$$\begin{array}{r} (C_{IN}) \\ A \\ \underline{+\ B} \\ S \\ (C_{OUT}) \end{array}$$





1-bit Binary Adder

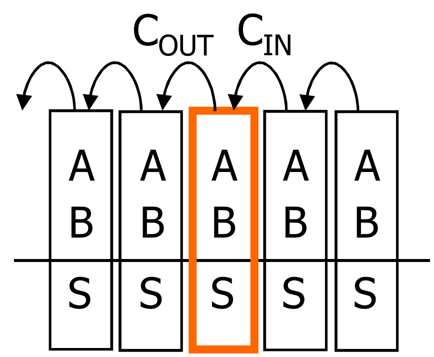
- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Derive an expression for S

$A' \cdot B' \cdot C_{IN}$

$A' \cdot B \cdot C_{IN}'$

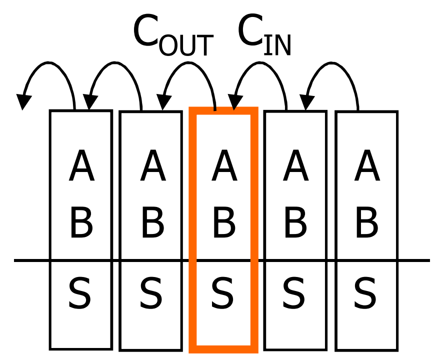
$A \cdot B' \cdot C_{IN}'$

$A \cdot B \cdot C_{IN}$

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Derive an expression for C_{OUT}

$A' \cdot B \cdot C_{IN}$

$A \cdot B' \cdot C_{IN}$

$A \cdot B \cdot C_{IN}'$

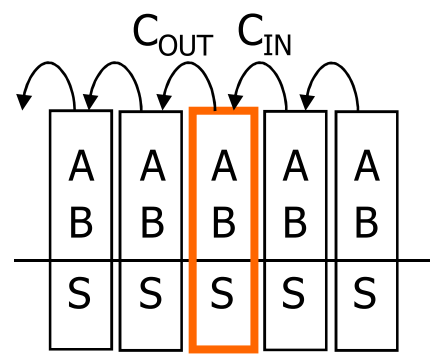
$A \cdot B \cdot C_{IN}$

$$C_{OUT} = A' \cdot B \cdot C_{IN} + A \cdot B' \cdot C_{IN} + A \cdot B \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$S = A' \bullet B' \bullet C_{IN} + A' \bullet B \bullet C_{IN}' + A \bullet B' \bullet C_{IN}' + A \bullet B \bullet C_{IN}$$

$$C_{OUT} = A' \bullet B \bullet C_{IN} + A \bullet B' \bullet C_{IN} + A \bullet B \bullet C_{IN}' + A \bullet B \bullet C_{IN}$$

Apply Theorems to Simplify Expressions

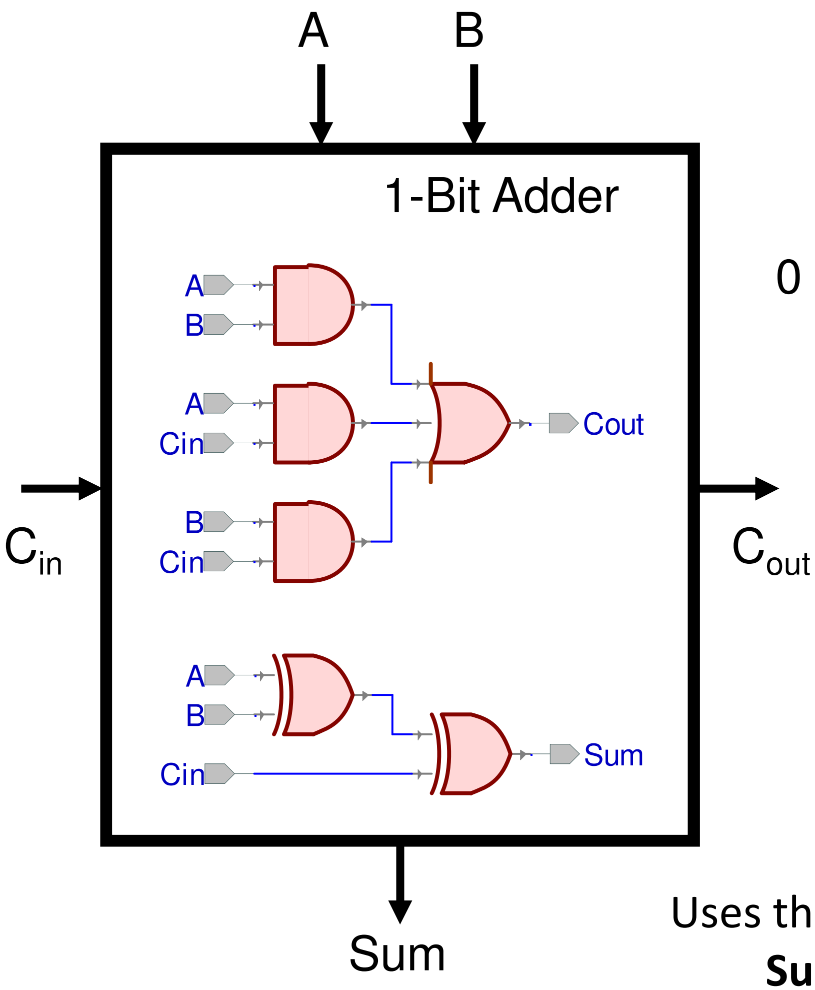
The theorems of Boolean algebra can simplify expressions

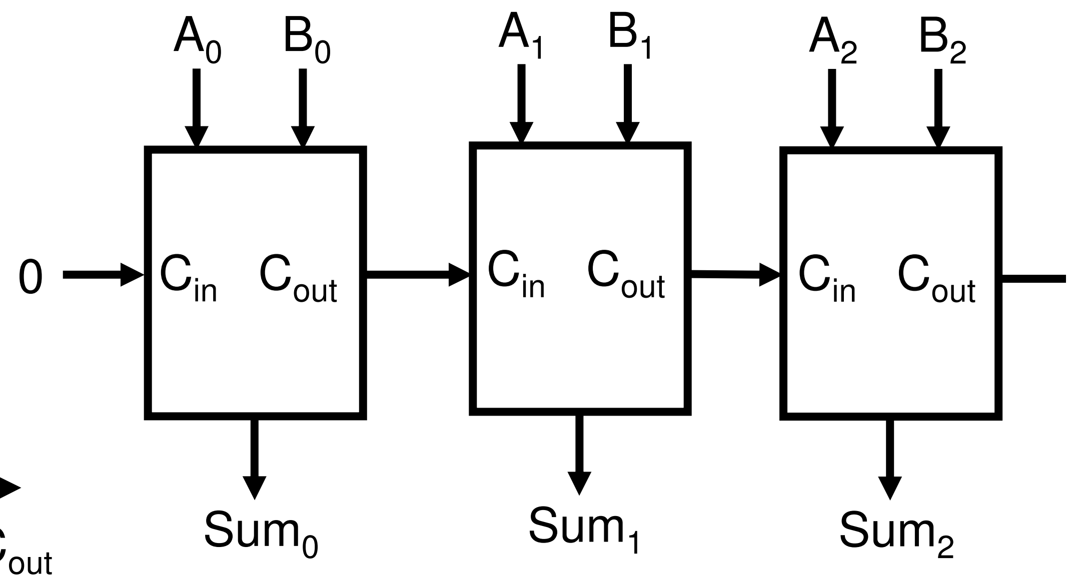
– **e.g., full adder's carry-out function**

$$
\begin{aligned}
\text{Cout} &= A' \, B \, Cin + A \, B' \, Cin + A \, B \, Cin' + A \, B \, Cin \\
&= A' \, B \, Cin + A \, B' \, Cin + A \, B \, Cin' + \boxed{A \, B \, Cin + A \, B \, Cin} \\
&= A' \, B \, Cin + A \, B \, Cin + A \, B' \, Cin + A \, B \, Cin' + A \, B \, Cin \\
&= (A' + A) \, B \, Cin + A \, B' \, Cin + A \, B \, Cin' + A \, B \, Cin \\
&= (1) \, B \, Cin + A \, B' \, Cin + A \, B \, Cin' + A \, B \, Cin \\
&= B \, Cin + A \, B' \, Cin + A \, B \, Cin' + \boxed{A \, B \, Cin + A \, B \, Cin} \\
&= B \, Cin + A \, B' \, Cin + A \, B \, Cin + A \, B \, Cin' + A \, B \, Cin \\
&= B \, Cin + A \, (B' + B) \, Cin + A \, B \, Cin' + A \, B \, Cin \\
&= B \, Cin + A \, (1) \, Cin + A \, B \, Cin' + A \, B \, Cin \\
&= B \, Cin + A \, Cin + A \, B \, (Cin' + Cin) \\
&= B \, Cin + A \, Cin + A \, B \, (1) \\
&= B \, Cin + A \, Cin + A \, B
\end{aligned}
$$

adding extra terms creates new factoring opportunities

A 2-bit Ripple-Carry Adder





Uses the fact that

$$\mathbf{Sum} = A' \bullet B' \bullet C_{IN} + A' \bullet B \bullet C_{IN}' + A \bullet B' \bullet C_{IN}' + A \bullet B \bullet C_{IN}$$

is equivalent to $\mathbf{Sum} = (A \oplus B) \oplus C_{IN}$

Mapping Truth Tables to Logic Gates

Given a truth table:

1. **Write the Boolean expression**
2. **Minimize the Boolean expression**
3. **Draw as gates**
4. **Map to available gates**

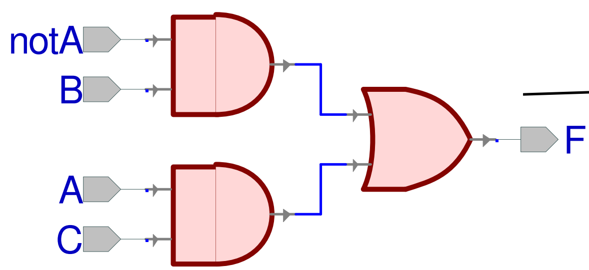
A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

1

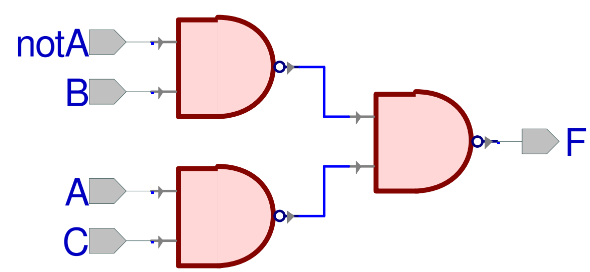
$$\begin{aligned} F &= A'BC' + A'BC + AB'C + ABC \\ &= A'B(C'+C) + AC(B'+B) \\ &= A'B + AC \end{aligned}$$

2

3



4



Canonical Forms

- Truth table is the unique signature of a Boolean Function
- The same truth table can have many gate realizations
  - We've seen this already
  - Depends on how good we are at Boolean simplification
- Canonical forms
  - Standard forms for a Boolean expression
  - We all come up with the same expression

Sum-of-Products Canonical Form

- AKA **Disjunctive Normal Form (DNF)**
- AKA **Minterm Expansion**

3 Add the minterms together

$$F = A'B'C + A'BC + AB'C + ABC' + ABC'$$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

1 Read T rows off truth table

2 Convert to Boolean Algebra

001 → $A'B'C$

011 → $A'BC$

101 → $AB'C$

110 → ABC'

111 → ABC

F

Sum-of-Products Canonical Form

Product term (or minterm)

- ANDed product of literals – input combination for which output is true
- each variable appears exactly once, true or inverted (but not both)

A	B	C	minterms
0	0	0	A′B′C′
0	0	1	A′B′C
0	1	0	A′BC′
0	1	1	A′BC
1	0	0	AB′C′
1	0	1	AB′C
1	1	0	ABC′
1	1	1	ABC

F in canonical form:

$F(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC$

canonical form ≠ minimal form

$$
\begin{aligned}
F(A, B, C) &= A'B'C + A'BC + AB'C + ABC + ABC' \\
&= (A'B' + A'B + AB' + AB)C + ABC' \\
&= ((A' + A)(B' + B))C + ABC' \\
&= C + ABC' \\
&= ABC' + C \\
&= AB + C
\end{aligned}
$$

Product-of-Sums Canonical Form

- AKA Conjunctive Normal Form (CNF)
- AKA Maxterm Expansion

4 Multiply the maxterms together

F =

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

1 Read F rows off truth table

2 Negate all bits

3 Convert to Boolean Algebra

F

Product-of-Sums Canonical Form

- AKA Conjunctive Normal Form (CNF)
- AKA Maxterm Expansion

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

1 Read F rows off truth table	2 Negate all bits	3 Convert to Boolean Algebra
000	111	$A + B + C$
010	101	$A + B' + C$
100	011	$A' + B + C$

F

4 Multiply the maxterms together

$$F = (A + B + C)(A + B' + C)(A' + B + C)$$

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F')' = F
- We know how to get a minterm expansion for F'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F')' = F
- We know how to get a minterm expansion for F'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Taking the complement of both sides...

$$(F')' = (A'B'C' + A'BC' + AB'C')'$$

And using DeMorgan/Comp....

$$F = (A'B'C')' \; (A'BC')' \; (AB'C')'$$

$$F = (A + B + C)(A + B' + C)(A' + B + C)$$

Product-of-Sums Canonical Form

Sum term (or maxterm)

- ORed sum of literals – input combination for which output is false
- each variable appears exactly once, true or inverted (but not both)

A	B	C	maxterms
0	0	0	$A+B+C$
0	0	1	$A+B+C'$
0	1	0	$A+B'+C$
0	1	1	$A+B'+C'$
1	0	0	$A'+B+C$
1	0	1	$A'+B+C'$
1	1	0	$A'+B'+C$
1	1	1	$A'+B'+C'$

F in canonical form:

$$F(A, B, C) = (A + B + C)\,(A + B' + C)\,(A' + B + C)$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= (A + B + C)\,(A + B' + C)\,(A' + B + C) \\ &= (A + B + C)\,(A + B' + C) \\ &\quad\ (A + B + C)\,(A' + B + C) \\ &= (A + C)\,(B + C) \end{aligned}$$

Predicate Logic

- **Propositional Logic**

  “If you take the high road and I take the low road then I’ll arrive in Scotland before you.”

- **Predicate Logic**

  “All positive integers x, y, and z satisfy $x^3 + y^3 \neq z^3$.”

Predicate Logic

- Propositional Logic
  - Allows us to analyze complex propositions in terms of their simpler constituent parts (a.k.a. atomic propositions) joined by connectives
- Predicate Logic
  - Lets us analyze them at a deeper level by expressing how those propositions depend on the objects they are talking about

Predicate Logic

Adds two key notions to propositional logic

- Predicates
- Quantifiers

Predicates

Predicate

– **A function that returns a truth value, e.g.,**

Cat(x) ::= “x is a cat”

Prime(x) ::= “x is prime”

HasTaken(x, y) ::= “student x has taken course y”

LessThan(x, y) ::= “x < y”

Sum(x, y, z) ::= “x + y = z”

GreaterThan5(x) ::= “x > 5”

HasNChars(s, n) ::= “string s has length n”

Predicates can have varying numbers of arguments and input types.

Domain of Discourse

For ease of use, we define one "type"/"domain" that we work over. This set of objects is called the "domain of discourse".

For each of the following, what might the domain be?

(1) "x is a cat", "x barks", "x ruined my couch"

(2) "x is prime", "$x = 0$", "$x < 0$", "x is a power of two"

(3) "student x has taken course y" "x is a pre-req for z"

Domain of Discourse

For ease of use, we define one "type"/"domain" that we work over. This set of objects is called the "**domain of discourse**".

For each of the following, what might the domain be?

(1) "x is a cat", "x barks", "x ruined my couch"

"mammals" or "sentient beings" or "cats and dogs" or ...

(2) "x is prime", "$x = 0$", "$x < 0$", "x is a power of two"

"numbers" or "integers" or "integers greater than 5" or ...

(3) "student x has taken course y" "x is a pre-req for z"

"students and courses" or "university entities" or ...

Quantifiers

We use *quantifiers* to talk about collections of objects.

$\forall x\ P(x)$

$P(x)$ is true **for every** x in the domain

read as "**for all x, P of x**"

$\exists x\ P(x)$

There is an x in the domain for which $P(x)$ is true

read as "**there exists x, P of x**"

Quantifiers

We use *quantifiers* to talk about collections of objects.

Universal Quantifier ("for all"): $\forall x\ P(x)$

$P(x)$ **is true for every** x **in the domain**

read as "for all x, P of x"

Examples: Are these true?

- $\forall x$ Odd(x)
- $\forall x$ LessThan5(x)

Quantifiers

We use *quantifiers* to talk about collections of objects.

Universal Quantifier ("for all"): $\forall x\ P(x)$

$P(x)$ **is true for every** x **in the domain**

read as "for all x, P of x"

Examples: **Are these true? It depends on the domain. For example:**

- $\forall x$ **Odd(x)**
- $\forall x$ **LessThan4(x)**

	{1, 3, -1, -27}	Integers	Odd Integers
$\forall x$ Odd(x)	True	False	True
$\forall x$ LessThan4(x)	True	False	False

Quantifiers

We use *quantifiers* to talk about collections of objects.

Existential Quantifier ("exists"): $\exists x\, P(x)$

There is an x in the domain for which P(x) is true

read as "there exists x, P of x"

Examples: Are these true?

- $\exists x$ Odd(x)
- $\exists x$ LessThan5(x)

Quantifiers

We use *quantifiers* to talk about collections of objects.

Existential Quantifier ("exists"): $\exists x\, P(x)$

There is an x in the domain for which $P(x)$ is true

read as "there exists x, P of x"

Examples: Are these true? It depends on the domain. For example:

	{1, 3, -1, -27}	Integers	Positive Multiples of 5
• $\exists$x Odd(x)	True	True	True
• $\exists$x LessThan4(x)	True	True	False

Statements with Quantifiers

Just like with propositional logic, we need to define variables (this time **predicates**) before we do anything else. We must also now define a **domain of discourse** before doing anything else.

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Determine the truth values of each of these statements:

$\exists x\ \text{Even}(x)$

$\forall x\ \text{Odd}(x)$

$\forall x\ (\text{Even}(x) \vee \text{Odd}(x))$

$\exists x\ (\text{Even}(x) \wedge \text{Odd}(x))$

$\forall x\ \text{Greater}(x+1, x)$

$\exists x\ (\text{Even}(x) \wedge \text{Prime}(x))$

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Determine the truth values of each of these statements:

$\exists x$ Even(x)	**T**	**e.g. 2, 4, 6, ...**
$\forall x$ Odd(x)	**F**	**e.g. 2, 4, 6, ...**
$\forall x$ (Even(x) $\vee$ Odd(x))	**T**	**every integer is either even or odd**
$\exists x$ (Even(x) $\wedge$ Odd(x))	**F**	**no integer is both even and odd**
$\forall x$ Greater(x+1, x)	**T**	**adding 1 makes a bigger number**
$\exists x$ (Even(x) $\wedge$ Prime(x))	**T**	**Even(2) is true and Prime(2) is true**

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\, \exists y\, \text{Greater}(y, x)$

$\forall x\, \exists y\, \text{Greater}(x, y)$

$\forall x\, \exists y\, (\text{Greater}(y, x) \wedge \text{Prime}(y))$

$\forall x\, (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

$\exists x\, \exists y\, (\text{Sum}(x, 2, y) \wedge \text{Prime}(x) \wedge \text{Prime}(y))$

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\, \exists y\, \text{Greater}(y, x)$

For every positive integer x, there is a positive integer y, such that y > x.

$\forall x\, \exists y\, \text{Greater}(x, y)$

For every positive integer x, there is a positive integer y, such that x > y.

$\forall x\, \exists y\, (\text{Greater}(y, x) \wedge \text{Prime}(y))$

For every positive integer x, there is a pos. int. y such that y > x and y is prime.

$\forall x\, (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

For each positive integer x, if x is prime, then x = 2 or x is odd.

$\exists x\, \exists y\, (\text{Sum}(x, 2, y) \wedge \text{Prime}(x) \wedge \text{Prime}(y))$

There exist positive integers x and y such that x + 2 = y and x and y are prime.

Statements with Quantifiers (Natural Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\, \exists y\, \text{Greater}(y, x)$

There is no greatest positive integer.

$\forall x\, \exists y\, \text{Greater}(x, y)$

There is no least positive integer.

$\forall x\, \exists y\, (\text{Greater}(y, x) \wedge \text{Prime}(y))$

For every positive integer there is a larger number that is prime.

$\forall x\, (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

Every prime number is either 2 or odd.

$\exists x\, \exists y\, (\text{Sum}(x, 2, y) \wedge \text{Prime}(x) \wedge \text{Prime}(y))$

There exist prime numbers that differ by two."

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= “x is a cat”
Red(x) ::= “x is red”
LikesTofu(x) ::= “x likes tofu”

“Red cats like tofu”

“Some red cats don’t like tofu”

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

"Red cats like tofu"

$$\forall x\ ((\mathrm{Red}(x) \land \mathrm{Cat}(x)) \rightarrow \mathrm{LikesTofu}(x))$$

"Some red cats don't like tofu"

$$\exists y\ ((\mathrm{Red}(y) \land \mathrm{Cat}(y)) \land \neg \mathrm{LikesTofu}(y))$$

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

When putting two predicates together like this, we use an "and".

"Red cats like tofu"

When restricting to a smaller domain in a "for all" we use implication.

When there's no leading quantification, it means "for all".

"Some red cats don't like tofu"

When restricting to a smaller domain in an "exists" we use and.

"Some" means "there exists".

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= "x is a purple fruit"

(*) $\forall x$ PurpleFruit(x) ("All fruits are purple")

What is the negation of (*)?

(a) "there exists a purple fruit"

(b) "there exists a non-purple fruit"

(c) "all fruits are not purple"

Try your intuition! Which one "feels" right?

Key Idea: In every domain, exactly one of a statement and its negation should be true.

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= “x is a purple fruit”

(*) $\forall x$ PurpleFruit(x) (“All fruits are purple”)

What is the negation of (*)?

(a) “there exists a purple fruit”

(b) “there exists a non-purple fruit”

(c) “all fruits are not purple”

Key Idea: In every domain, exactly one of a statement and its negation should be true.

Domain of Discourse
{plum}

Domain of Discourse
{apple}

Domain of Discourse
{plum, apple}

The only choice that ensures exactly one of the statement and its negation is (b).

De Morgan's Laws for Quantifiers

$$\boxed{\begin{array}{c} \neg\forall x\ P(x) \equiv \exists x \neg\ P(x) \\ \neg\ \exists x\ P(x) \equiv \forall x \neg\ P(x) \end{array}}$$

De Morgan's Laws for Quantifiers

$$\boxed{\begin{array}{c} \neg\forall x\, P(x) \equiv \exists x\, \neg P(x) \\ \neg\exists x\, P(x) \equiv \forall x\, \neg P(x) \end{array}}$$

"There is no largest integer"

$$\begin{aligned} & \neg\exists x\, \forall y\, (x \geq y) \\ \equiv\ & \forall x\, \neg\forall y\, (x \geq y) \\ \equiv\ & \forall x\, \exists y\, \neg(x \geq y) \\ \equiv\ & \forall x\, \exists y\, (x < y) \end{aligned}$$

"For every integer there is a larger integer"

Scope of Quantifiers

$\exists x\ (P(x) \land Q(x))$ **vs.** $\exists x\, P(x) \land \exists x\, Q(x)$

Scope of Quantifiers

$\exists x\ (P(x) \wedge Q(x))$ **vs.** $\exists x\, P(x) \wedge \exists x\, Q(x)$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different *x*'s.

Scope of Quantifiers

Example: $\text{NotLargest}(x) \equiv \exists\, y\ \text{Greater}\,(y, x)$
$\equiv \exists\, z\ \text{Greater}\,(z, x)$

truth value:

doesn't depend on y **or** z **"bound variables"**

does depend on x **"free variable"**

quantifiers only act on free variables of the formula they quantify

$$\forall\, x\ (\exists\, y\ (P(x,y) \rightarrow \forall\, x\ Q(y, x)))$$

Quantifier "Style"

$$\forall x(\exists y\ (P(x,y) \rightarrow \forall\ x\ Q(y, x)))$$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…