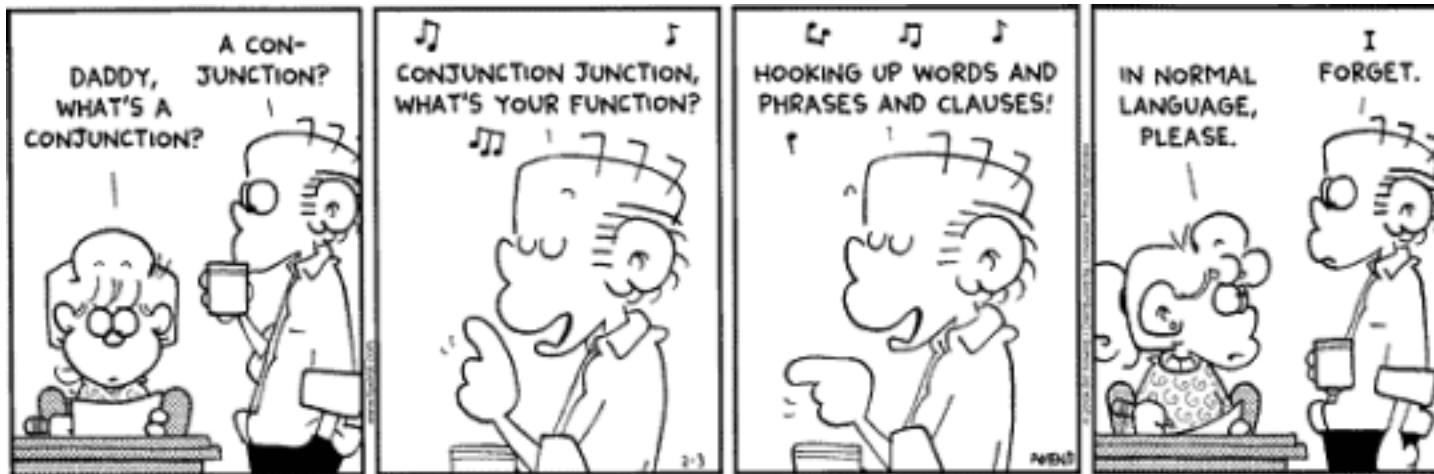


# CSE 311: Foundations of Computing

## Lecture 2: More Logic, Equivalence & Digital Circuits



- Everyone who petitioned should have been signed up by now
- Canvas will be published later today
- 1<sup>st</sup> Homework out today
- Section tomorrow.

# Last class: Some Connectives & Truth Tables

---

Negation (not)

$p$	$\neg p$
T	F
F	T

Conjunction (and)

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (or)

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

$p$	$q$	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

# Last class: Implication

---

*“If it’s raining, then I have my umbrella”*

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$$p \rightarrow q$$

---

**(1) “I have collected all 151 Pokémon if I am a Pokémon master”**

**(2) “I have collected all 151 Pokémon only if I am a Pokémon master”**

**These sentences are implications in opposite directions:**

$$p \rightarrow q$$

---

(1) *“I have collected all 151 Pokémon if I am a Pokémon master”*

(2) *“I have collected all 151 Pokémon only if I am a Pokémon master”*

These sentences are implications in opposite directions:

(1) **“Pokémon masters have all 151 Pokémon”**

(2) **“People who have 151 Pokémon are Pokémon masters”**

So, the implications are:

(1) *If I am a Pokémon master, then I have collected all 151 Pokémon.*

(2) *If I have collected all 151 Pokémon, then I am a Pokémon master.*

$$p \rightarrow q$$

---

## Implication:

- $p$  implies  $q$
- whenever  $p$  is true  $q$  must be true
- if  $p$  then  $q$
- $q$  if  $p$
- $p$  is sufficient for  $q$
- $p$  only if  $q$
- $q$  is necessary for  $p$

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

# Biconditional: $p \leftrightarrow q$

---

- $p$  iff  $q$       "*if and only if*"
- $p$  is equivalent to  $q$
- $p$  implies  $q$  and  $q$  implies  $p$
- $p$  is necessary and sufficient for  $q$

$p$	$q$	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

# Biconditional: $p \leftrightarrow q$

---

- $p$  iff  $q$
- $p$  is equivalent to  $q$
- $p$  implies  $q$  and  $q$  implies  $p$
- $p$  is necessary and sufficient for  $q$

$p$	$q$	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T



# Last class: Using Logical Connectives

---

Measles:

“You can get measles”

Mumps:

“You can get mumps”

MMR:

“You had the MMR vaccine”

“You can get measles and mumps if you didn’t have the MMR vaccine, but if you had the MMR vaccine then you can’t get either.”



$((\text{Measles and Mumps}) \text{ if not MMR}) \text{ and } (\text{if MMR then not } (\text{Measles or Mumps}))$



$((\text{Measles} \wedge \text{Mumps}) \text{ if } \neg \text{MMR}) \wedge (\text{if MMR then } \neg(\text{Measles} \vee \text{Mumps}))$

# Understanding the Vaccine Sentence

---

**“You can get measles and mumps if you didn’t have the MMR vaccine, but if you had the MMR vaccine you can’t get either.”**



$((\text{Measles} \wedge \text{Mumps}) \text{ if } \neg \text{MMR}) \wedge (\text{if MMR then } \neg(\text{Measles} \vee \text{Mumps}))$



$(\neg \text{MMR} \rightarrow (\text{Measles} \wedge \text{Mumps})) \wedge (\text{MMR} \rightarrow \neg(\text{Measles} \vee \text{Mumps}))$

# Understanding the Vaccine Sentence

---

“You can get measles and mumps if you didn’t have the MMR vaccine, but if you had the MMR vaccine you can’t get either.”

$((\text{Measles} \wedge \text{Mumps}) \text{ if } \neg \text{MMR}) \wedge (\text{if MMR then } \neg(\text{Measles} \vee \text{Mumps}))$

$(\neg \text{MMR} \rightarrow (\text{Measles} \wedge \text{Mumps})) \wedge (\text{MMR} \rightarrow \neg(\text{Measles} \vee \text{Mumps}))$

Define shorthand ...

$p$  : MMR

$q$  : Measles

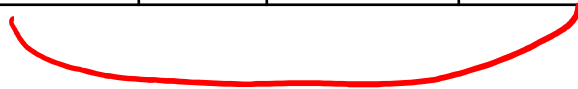
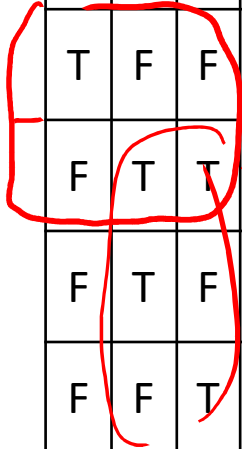
$r$  : Mumps

$(\neg p \rightarrow (q \wedge r)) \wedge (p \rightarrow \neg(q \vee r))$

# Analyzing the Vaccine Sentence with a Truth Table

---

$p$	$q$	$r$	$\neg p$	$q \wedge r$	$\neg p \rightarrow (q \wedge r)$	$q \vee r$	$\neg(q \vee r)$	$p \rightarrow \neg(q \vee r)$	$(\neg p \rightarrow (q \wedge r)) \wedge (p \rightarrow \neg(q \vee r))$
T	T	T	F	T	T	T	F	F	F
T	T	F	F	F	T	T	F	F	F
T	F	T	F	F	T	T	F	F	F
T	F	F	F	F	T	F	T	T	T
F	T	T	T	T	T	T	F	T	T
F	T	F	T	F	F	T	F	T	F
F	F	T	T	F	F	T	F	T	F
F	F	F	T	F	F	F	T	T	F



# Analyzing the Vaccine Sentence with a Truth Table

---

$p$	$q$	$r$	$\neg p$	$q \wedge r$	$\neg p \rightarrow (q \wedge r)$	$q \vee r$	$\neg(q \vee r)$	$p \rightarrow \neg(q \vee r)$	$(\neg p \rightarrow (q \wedge r)) \wedge (p \rightarrow \neg(q \vee r))$
T	T	T	F	T	T	T	F	F	F
T	T	F	F	F	T	T	F	F	F
T	F	T	F	F	T	T	F	F	F
T	F	F	F	F	T	F	T	T	T
F	T	T	T	T	T	T	F	T	T
F	T	F	T	F	F	T	F	T	F
F	F	T	T	F	F	T	F	T	F
F	F	F	T	F	F	F	T	T	F

# Converse, Contrapositive

---

**Implication:**

$$p \rightarrow q$$

**Converse:**

$$q \rightarrow p$$

**Contrapositive:**

$$\neg q \rightarrow \neg p$$

**Inverse:**

$$\neg p \rightarrow \neg q$$

Consider

$p$ :  $x$  is divisible by 2

$q$ :  $x$  is divisible by 4

$p \rightarrow q$	
$q \rightarrow p$	
$\neg q \rightarrow \neg p$	
$\neg p \rightarrow \neg q$	

# Converse, Contrapositive

---

**Implication:**

$$p \rightarrow q$$

**Converse:**

$$q \rightarrow p$$

**Contrapositive:**

$$\neg q \rightarrow \neg p$$

**Inverse:**

$$\neg p \rightarrow \neg q$$

Consider

**$p$ :  $x$  is divisible by 2**

**$q$ :  $x$  is divisible by 4**

$p \rightarrow q$	
$q \rightarrow p$	
$\neg q \rightarrow \neg p$	
$\neg p \rightarrow \neg q$	

	Divisible By 2	Not Divisible By 2
Divisible By 4		
Not Divisible By 4		

# Converse, Contrapositive

---

Implication:

$$p \rightarrow q$$

Converse:

$$q \rightarrow p$$

Contrapositive:

$$\neg q \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg q$$

Consider

$p$ :  $x$  is divisible by 2

$q$ :  $x$  is divisible by 4

$p \rightarrow q$	F
$q \rightarrow p$	T
$\neg q \rightarrow \neg p$	F
$\neg p \rightarrow \neg q$	T

	Divisible By 2	Not Divisible By 2
Divisible By 4	4,8,12,...	Impossible
Not Divisible By 4	2,6,10,...	1,3,5,...



# Converse, Contrapositive

---

Implication:

$$p \rightarrow q$$

Converse:

$$q \rightarrow p$$


Contrapositive:

$$\neg q \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg q$$

How do these relate to each other?



$p$	$q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	<u><math>\neg p \rightarrow \neg q</math></u>	$\neg q \rightarrow \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

# Converse, Contrapositive

---

Implication:

$$p \rightarrow q$$

Converse:

$$q \rightarrow p$$

Contrapositive:

$$\neg q \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg q$$

An **implication** and its **contrapositive**  
have the same truth value!

$p$	$q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

# Tautologies!

---

**Terminology:** A compound proposition is a...

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$$p \vee \neg p$$

Tautology

$$p \oplus p$$

Contradiction

$$(p \rightarrow q) \wedge p$$

Contingency

p and q are T  
q is T  
p is false q is T F

# Tautologies!

---

**Terminology:** A compound proposition is a...

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$$p \vee \neg p$$

This is a tautology. It's called the "law of the excluded middle. If  $p$  is true, then  $p \vee \neg p$  is true. If  $p$  is false, then  $p \vee \neg p$  is true.

$$p \oplus p$$

This is a contradiction. It's always false no matter what truth value  $p$  takes on.

$$(p \rightarrow q) \wedge p$$

This is a contingency. When  $p=T, q=T, (T \rightarrow T) \wedge T$  is true.  
When  $p=T, q=F, (T \rightarrow F) \wedge T$  is false.

# Logical Equivalence

---

**A = B** means **A** and **B** are identical “strings”:

–  $p \wedge q = p \wedge q$

–  $p \wedge q \neq q \wedge p$

# Logical Equivalence

---

**A = B** means **A** and **B** are identical “strings”:

–  $p \wedge q = p \wedge q$

These are equal, because they are character-for-character identical.

–  $p \wedge q \neq q \wedge p$

These are NOT equal, because they are different sequences of characters. They “mean” the same thing though.

**A ≡ B** means **A** and **B** have identical truth values:

–  $p \wedge q \equiv p \wedge q$

–  $p \wedge q \equiv q \wedge p$

–  $p \wedge q \neq q \vee p$

# Logical Equivalence

---

**A = B** means **A** and **B** are identical “strings”:

–  $p \wedge q = p \wedge q$

These are equal, because they are character-for-character identical.

–  $p \wedge q \neq q \wedge p$

These are NOT equal, because they are different sequences of characters. They “mean” the same thing though.

**A ≡ B** means **A** and **B** have identical truth values:

–  $p \wedge q \equiv p \wedge q$

Two formulas that are equal also are equivalent.

–  $p \wedge q \equiv q \wedge p$

These two formulas have the same truth table!

–  $p \wedge q \neq q \vee p$

When  $p=T$  and  $q=F$ ,  $p \wedge q$  is false, but  $p \vee q$  is true!

## $A \leftrightarrow B$ vs. $A \equiv B$

---

$A \equiv B$  is an assertion over all possible truth values that  $A$  and  $B$  always have the same truth values.

$A \leftrightarrow B$  is a proposition that may be true or false depending on the truth values of the variables in  $A$  and  $B$ .

$A \equiv B$  and  $(A \leftrightarrow B) \equiv T$  have the same meaning.

$A \leftrightarrow B$  is a tautology



# De Morgan's Laws

---

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$



Negate the statement:

**“My code compiles or there is a bug.”**

To negate the statement, ask “when is the original statement false”.

# De Morgan's Laws

---

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Negate the statement:

“My code compiles or there is a bug.”

To negate the statement, ask “when is the original statement false”.

It's false when not(my code compiles) AND not(there is a bug).

Translating back into English, we get:

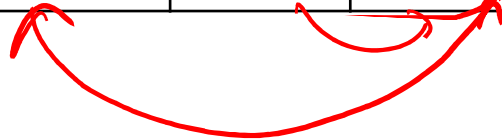
My code doesn't compile and there is not a bug.

# De Morgan's Laws

---

Example:  $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$

$p$	$q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
T	T	F	F	F	T	F	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T



# De Morgan's Laws

---

**Example:**  $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$

$p$	$q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
T	T	F	F	F	T	F	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T

# De Morgan's Laws

---

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$\neg$   $\neg$  (front == null)  $\wedge$

```
if (!(front != null  $\&\&$  value > front.data))
    front = new ListNode(value, front);
else {
    ListNode current = front;
    while (current.next != null  $\&\&$  current.next.data < value))
        current = current.next;
    current.next = new ListNode(value, current.next);
}
```

$(\text{front} == \text{null}) \parallel (\text{value} \leq \text{front.data})$

# De Morgan's Laws

---

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

```
!(front != null && value > front.data)
```

≡

```
front == null || value <= front.data
```

**You've been using these for a while!**

# Law of Implication

---

$$p \rightarrow q \equiv \neg p \vee q$$

$p$	$q$	$p \rightarrow q$	$\neg p$	$\neg p \vee q$	$p \rightarrow q \leftrightarrow \neg p \vee q$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

# Law of Implication

---

$$p \rightarrow q \equiv \neg p \vee q$$

$p$	$q$	$p \rightarrow q$	$\neg p$	$\neg p \vee q$	$p \rightarrow q \leftrightarrow \neg p \vee q$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T



## Some Equivalences Related to Implication

---

$$\begin{array}{lcl} p \rightarrow q & \equiv & \neg p \vee q \quad \checkmark \text{ Implication} \\ p \rightarrow q & \equiv & \neg q \rightarrow \neg p \quad \checkmark \text{ Contrapositive} \\ p \leftrightarrow q & \equiv & \underline{(p \rightarrow q) \wedge (q \rightarrow p)} \\ p \leftrightarrow q & \equiv & \neg p \leftrightarrow \neg q \end{array}$$

# Properties of Logical Connectives

---

We will always give  
you this list!

- **Identity**

- $p \wedge T \equiv p$

- $p \vee F \equiv p$

- **Domination**

- $p \vee T \equiv T$

- $p \wedge F \equiv F$

- **Idempotent**

- $p \vee p \equiv p$

- $p \wedge p \equiv p$

- **Commutative**

- $p \vee q \equiv q \vee p$

- $p \wedge q \equiv q \wedge p$

- **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$

- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- **Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **Absorption**

- $p \vee (p \wedge q) \equiv p$

- $p \wedge (p \vee q) \equiv p$

- **Negation**

- $p \vee \neg p \equiv T$

- $p \wedge \neg p \equiv F$

# Digital Circuits

---

## Computing With Logic

- **T** corresponds to **1** or “high” voltage
- **F** corresponds to **0** or “low” voltage

## Gates

- Take inputs and produce outputs (functions)
- Several kinds of gates
- Correspond to propositional connectives (most of them)

# And Gate

---

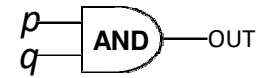
**AND Connective**

vs.

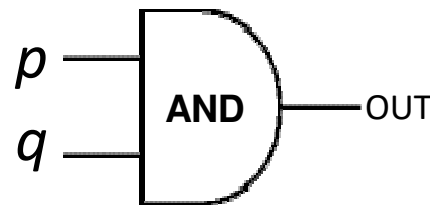
**AND Gate**

$p \wedge q$

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F



$p$	$q$	OUT
1	1	1
1	0	0
0	1	0
0	0	0



“block looks like D of AND”

# Or Gate

---

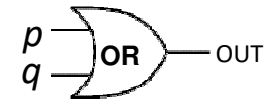
**OR Connective**

vs.

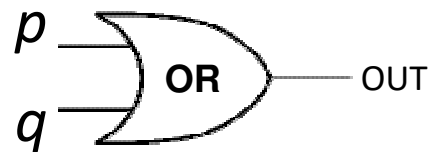
**OR Gate**

$p \vee q$

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F



$p$	$q$	OUT
1	1	1
1	0	1
0	1	1
0	0	0



“arrowhead block looks like V”

# Not Gates

---

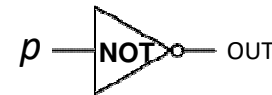
## NOT Connective

$$\neg p$$

$p$	$\neg p$
T	F
F	T

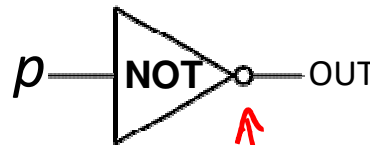
vs.

## NOT Gate



Also called  
*inverter*

$p$	OUT
1	0
0	1

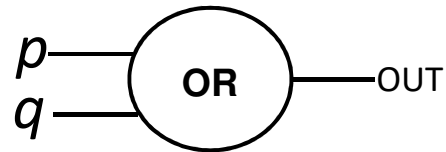
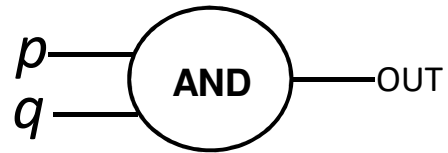


*bubble*  $\Rightarrow$  *most important part.*

# Blobs are Okay!

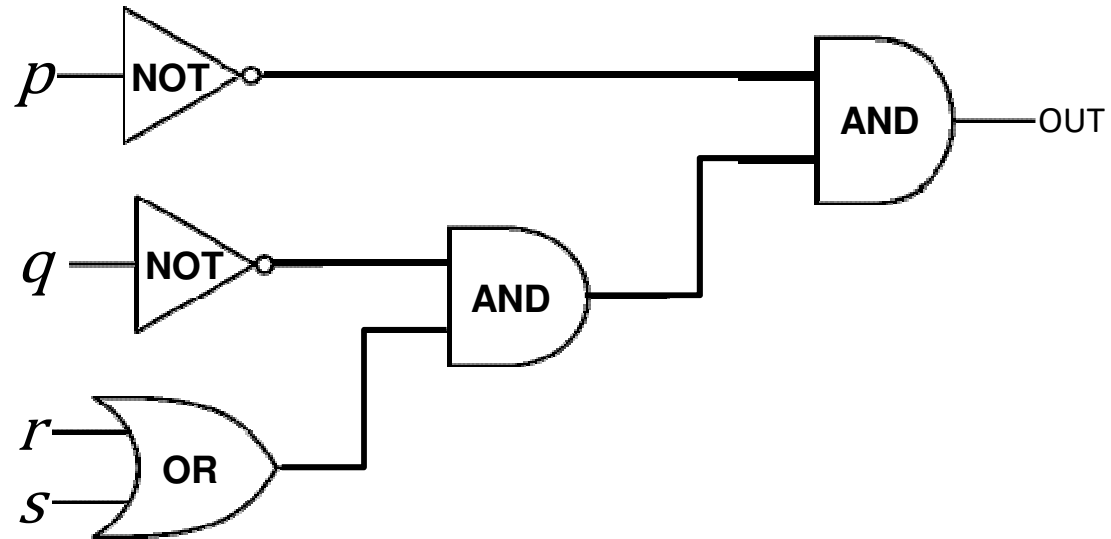
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You may write gates using blobs instead of shapes!



# Combinational Logic Circuits

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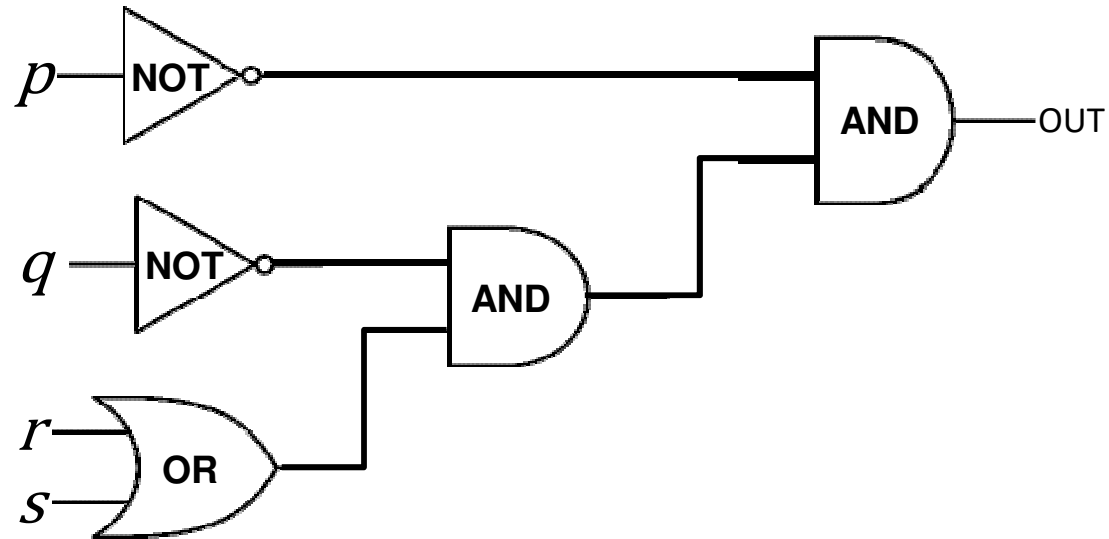


**Values get sent along wires connecting gates**



# Combinational Logic Circuits

---

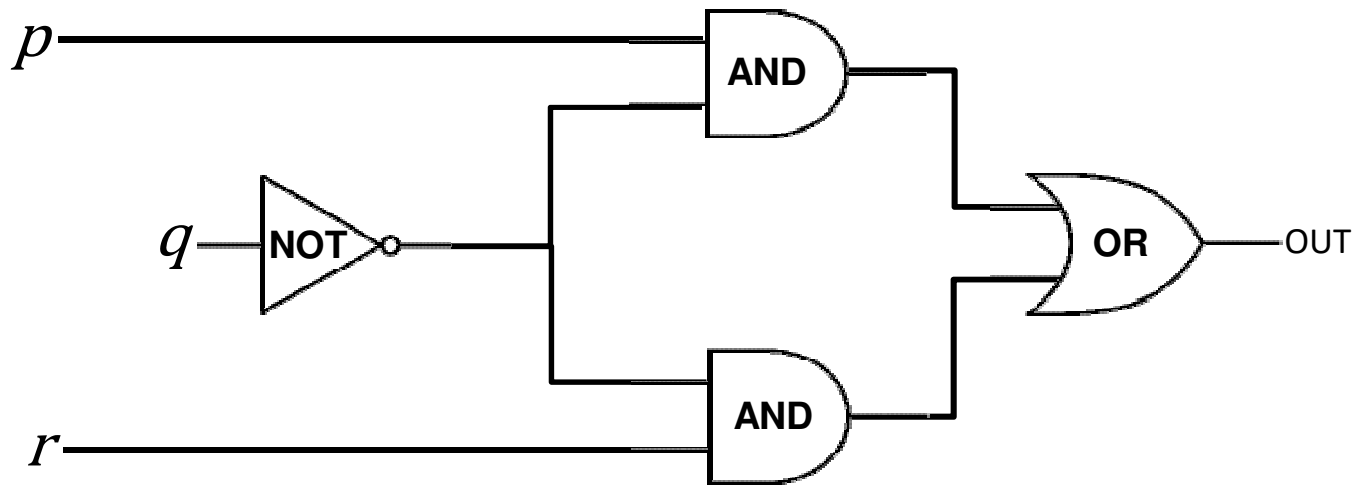


**Values get sent along wires connecting gates**

$$\neg p \wedge (\neg q \wedge (r \vee s))$$

# Combinational Logic Circuits

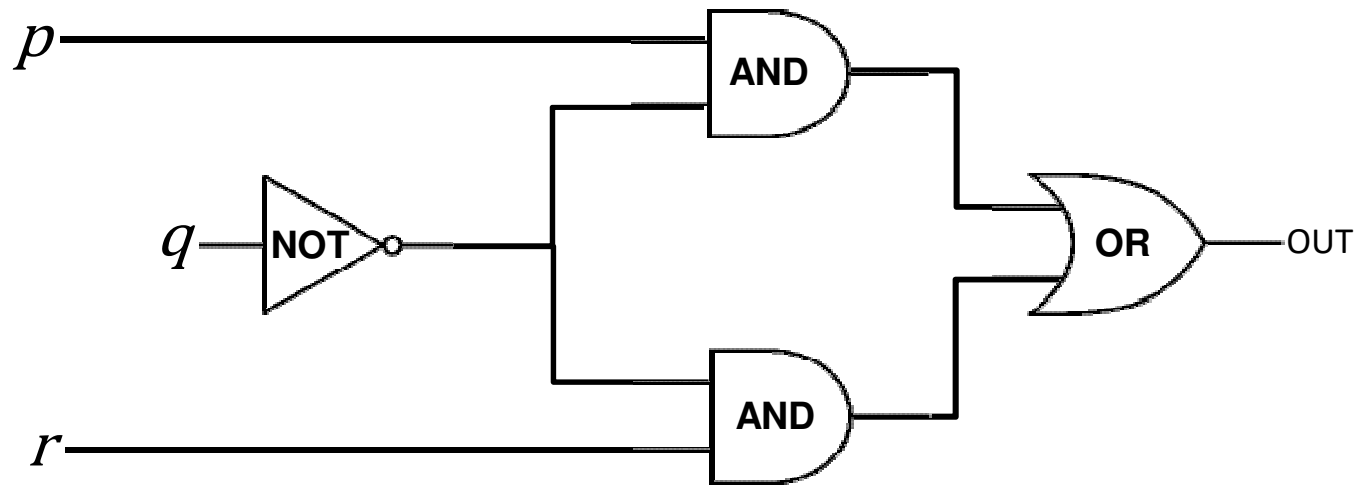
---



**Wires can send one value to multiple gates!**

# Combinational Logic Circuits

---



**Wires can send one value to multiple gates!**

$$(p \wedge \neg q) \vee (\neg q \wedge r)$$

# Computing Equivalence

---

Describe an algorithm for computing if two logical expressions/circuits are equivalent.

**What is the run time of the algorithm?**

Compute the entire truth table for both of them!

There are  $2^n$  entries in the column for  $n$  variables.

## Some Familiar Properties of Arithmetic

---

- $x + y = y + x$  (Commutativity)
- $x \cdot (y + z) = x \cdot y + x \cdot z$  (Distributivity)
- $(x + y) + z = x + (y + z)$  (Associativity)

# Understanding Connectives

---

- **Reflect basic rules of reasoning and logic**
- **Allow manipulation of logical formulas**
  - Simplification
  - Testing for equivalence
- **Applications**
  - Query optimization
  - Search optimization and caching
  - Artificial Intelligence
  - Program verification