

CSE 311: Foundations of Computing I

Section : Induction and Strong Induction Solutions

1. Induction

(a) Prove that $9 \mid n^3 + (n + 1)^3 + (n + 2)^3$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be " $9 \mid n^3 + (n + 1)^3 + (n + 2)^3$ ". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case ($n = 2$): $2^3 + (2 + 1)^3 + (2 + 2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2 + 1)^3 + (2 + 2)^3$, so $P(2)$ holds.

Induction Hypothesis: Assume that $9 \mid j^3 + (j + 1)^3 + (j + 2)^3$ for some arbitrary integer $j > 1$. Note that this is equivalent to assuming that $j^3 + (j + 1)^3 + (j + 2)^3 = 9k$ for some integer k .

Induction Step: Goal: Show $9 \mid (j + 1)^3 + (j + 2)^3 + (j + 3)^3$

$$\begin{aligned}(j + 1)^3 + (j + 2)^3 + (j + 3)^3 &= (j + 3)^3 + 9k - j^3 \quad \text{[Induction Hypothesis]} \\ &= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\ &= 9j^2 + 27j + 27 + 9k \\ &= 9(j^2 + 3j + 3 + k)\end{aligned}$$

So $9 \mid (j + 1)^3 + (j + 2)^3 + (j + 3)^3$, so $P(j) \rightarrow P(j + 1)$ for an arbitrary integer $j > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

(b) Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Solution:

Let $P(n)$ be " $6n + 6 < 2^n$ ". We will prove $P(n)$ for all integers $n \geq 6$ by induction.

Base Case ($n = 6$): $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

Induction Hypothesis: Assume that $6j + 6 < 2^j$ for some arbitrary integer $j \geq 6$.

Induction Step: Goal: Show $6(j + 1) + 6 < 2^{j+1}$

$$\begin{aligned}6(j + 1) + 6 &= 6j + 6 + 6 \\ &< 2^j + 6 && \text{[Induction Hypothesis]} \\ &< 2^j + 2^j && \text{[Since } 2^j > 6, \text{ since } j \geq 6\text{]} \\ &< 2 \cdot 2^j \\ &< 2^{j+1}\end{aligned}$$

So $P(j) \rightarrow P(j + 1)$ for an arbitrary integer $j \geq 6$.

Conclusion: $P(n)$ holds for all integers $n \geq 6$ by induction.

(c) Define

$$H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for $n \in \mathbb{N}$.

Solution:

We define H_i more formally as $\sum_{k=1}^i \frac{1}{k}$. Let $P(n)$ be " $H_{2^n} \geq 1 + \frac{n}{2}$ ". We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

Base Case ($n = 0$): $H_{2^0} = H_1 = \sum_{k=1}^1 \frac{1}{k} = 1 \geq 1 + \frac{0}{2}$, so $P(0)$ holds.

Induction Hypothesis: Assume that $H_{2^j} \geq 1 + \frac{j}{2}$ for some arbitrary integer $j \in \mathbb{N}$.

Induction Step: Goal: Show $H_{2^{j+1}} \geq 1 + \frac{j+1}{2}$

$$\begin{aligned} H_{2^{j+1}} &= \sum_{k=1}^{2^{j+1}} \frac{1}{k} \\ &= \sum_{k=1}^{2^j} \frac{1}{k} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \\ &\geq 1 + \frac{j}{2} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} && \text{[Induction Hypothesis]} \\ &\geq 1 + \frac{j}{2} + 2^j \cdot \frac{1}{2^{j+1}} && \text{[There are } 2^j \text{ terms in } [2^j + 1, 2^{j+1}] \text{ and each is at least } \frac{1}{2^{j+1}} \text{]} \\ &\geq 1 + \frac{j}{2} + \frac{2^j}{2^{j+1}} \\ &\geq 1 + \frac{j}{2} + \frac{1}{2} \geq 1 + \frac{j+1}{2} \end{aligned}$$

So $P(j) \rightarrow P(j+1)$ for an arbitrary integer $j \in \mathbb{N}$.

Conclusion: $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.

2. Strong Induction

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f :

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f(n) &= 2f(n-1) - f(n-2) \text{ for } n \geq 2 \end{aligned}$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n .

Solution:

Let $P(n)$ be " $f(n) = n$ ". We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on n .

Base Case $n = 0$: $f(0) = 0$ by definition.

Induction Hypothesis: Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every integer j with $0 \leq j \leq k$.

Induction Step: We show $P(k+1)$: We have two cases depending on whether $k+1 = 1$ or $k+1 \geq 2$. When $k+1 = 1$, we have $f(k+1) = f(1) = 1 = k+1$ which is what we needed to prove. When $k+1 \geq 2$,

we have

$$\begin{aligned}f(k+1) &= 2f(k) - f(k-1) \\ &= 2k - (k-1) \\ &= k+1\end{aligned}$$

[Definition of f]
[Induction Hypothesis]
[Algebra]

Therefore $P(k+1)$ is true in all cases.

Therefore, by induction $f(n) = n$ for all $n \in \mathbb{N}$.