

CSE 311: Foundations of Computing I

Section : Sets and Modular Arithmetic Solutions

1. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say .

(a) $A = \{1, 2, 3, 2\}$

Solution:

3

(b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \dots\}$

Solution:

$$\begin{aligned} B &= \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \dots\} \\ &= \{\{\}, \{\{\}\}, \{\{\}\}, \{\{\}\}, \dots\} \\ &= \{\emptyset, \{\emptyset\}\} \end{aligned}$$

So, there are two elements in B .

(c) $C = A \times (B \cup \{7\})$

Solution:

$C = \{1, 2, 3\} \times \{\emptyset, \{\emptyset\}, 7\} = \{(a, b) \mid a \in \{1, 2, 3\}, b \in \{\emptyset, \{\emptyset\}, 7\}\}$. It follows that there are $3 \times 3 = 9$ elements in C .

(d) $D = \emptyset$

Solution:

0.

(e) $E = \{\emptyset\}$

Solution:

1.

(f) $F = \mathcal{P}(\{\emptyset\})$

Solution:

$2^1 = 2$. The elements are $F = \{\emptyset, \{\emptyset\}\}$.

2. Set = Set

Prove the following set identities.

(a) Let the universal set be \mathcal{U} . Prove $A \cap \overline{B} \subseteq A \setminus B$ for any sets A, B .

Solution:

Let x be arbitrary.

$$\begin{aligned}x \in A \cap \overline{B} &\rightarrow x \in A \wedge x \in \overline{B} \quad [\text{Definition of } \cap] \\&\rightarrow x \in A \wedge x \notin B \quad [\text{Definition of } \overline{B}] \\&\rightarrow x \in A \setminus B \quad [\text{Definition of } \setminus]\end{aligned}$$

Thus, since $x \in A \cap \overline{B} \rightarrow x \in A \setminus B$, it follows that $A \cap \overline{B} \subseteq A \setminus B$, by definition of subset.

- (b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D .

Solution:

Let x be an arbitrary element of $(A \cap B) \times C$. Then, by definition of Cartesian product, x must be of the form (y, z) where $y \in A \cap B$ and $z \in C$. Since $y \in A \cap B$ by definition of \cap , $y \in A$ and $y \in B$; in particular, all we care about is that $y \in A$. Since $z \in C$, by definition of \cup , we also have $z \in C \cup D$. Therefore since $y \in A$ and $z \in C \cup D$, by definition of Cartesian product we have $x = (y, z) \in A \times (C \cup D)$.

Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$ as required.

3. Modular Arithmetic

- (a) Prove that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

Solution:

Suppose that $a \mid b$ and $b \mid a$, where a, b are integers. By the definition of divides, we have $a \neq 0$, $b \neq 0$ and $b = ka, a = jb$ for some integers k, j . Combining these equations, we see that $a = j(ka)$.

Then, dividing both sides by a , we get $1 = jk$. So, $\frac{1}{j} = k$. Note that j and k are integers, which is only possible if $j, k \in \{1, -1\}$. It follows that $b = -a$ or $b = a$.

- (b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Solution:

Suppose $n \mid m$ with $n, m > 1$, and $a \equiv b \pmod{m}$. By definition of divides, we have $m = kn$ for some $k \in \mathbb{Z}$. By definition of congruence, we have $m \mid a - b$, which means that $a - b = mj$ for some $j \in \mathbb{Z}$. Combining the two equations, we see that $a - b = (knj) = n(kj)$. By definition of congruence, we have $a \equiv b \pmod{n}$, as required.

4. Casting Out Nines

Let $n \in \mathbb{N}$. Prove that if $n \equiv 0 \pmod{9}$, then the sum of the digits of n is a multiple of 9.

You may use without proof that for any integer $i \geq 1$, $a \equiv b \pmod{m} \rightarrow a^i \equiv b^i \pmod{m}$. (This is trivial for $i = 1$ and we already proved that it holds when $i = 2$. We will later prove this for all larger i by mathematical induction.)

Solution:

Write $n = (d_m d_{m-1} \cdots d_1 d_0)_{10}$ where d_0, \dots, d_m are the base-10 digits of n . Then $n = \sum_{i=0}^m d_i 10^i$ by definition. We show that $n \equiv \sum_{i=0}^m d_i \pmod{9}$: Now $10 \equiv 1 \pmod{9}$ and so by the fact we know that for every $i \geq 1$, $10^i \equiv 1^i \pmod{9}$ which is just $10^i \equiv 1 \pmod{9}$. We also have $10^0 = 1$. Therefore, for any

$i = 0, \dots, m$ by the multiplicative property modulo 9 we have $d_i 10^i \equiv d_i \pmod{9}$. We then repeatedly apply the sum property modulo 9 to derive that $\sum_{i=0}^m d_i 10^i \equiv \sum_{i=0}^m d_i \pmod{9}$. The left-hand quantity is just n by definition so we have $n \equiv \sum_{i=0}^m d_i \pmod{9}$.

In particular, since $n \equiv 0 \pmod{9}$ by assumption, we have $\sum_{i=0}^m d_i \equiv 0 \pmod{9}$ and hence 9 divides the sum of the digits of n which is what we wanted to prove.