CSE 311: Foundations of Computing I

Homework 4 (due Wednesday, April 25 at 11:59 PM)

Directions: *Write up carefully argued solutions to the following problems. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use results from lecture, the theorems handout, and previous homeworks without proof.*

1. Don't Cross Me (16 points)

Let the domain of discourse be the real numbers ($\mathbb{R}$). We define the predicate OnLine(a, b, x, y) to be true iff (x, y) lies on the line with slope a and intercept b (i.e., iff $ax + b = y$) and the predicate OnCircle(u, v, r, x, y) to be true iff $r > 0$ and (x, y) lies on the circle of radius r with center at (u, v).

Give an English proof of the following claim:

$$\forall u\, \forall v\, \forall r\, ((r > 0) \rightarrow \exists a\, \exists b\, \exists x_1\, \exists y_1\, \exists x_2\, \exists y_2\, (((x_1 \neq x_2) \vee (y_1 \neq y_2)) \wedge \text{OnCircle}(u, v, r, x_1, y_1) \wedge \text{OnCircle}(u, v, r, x_2, y_2) \wedge \text{OnLine}(a, b, x_1, y_1) \wedge \text{OnLine}(a, b, x_2, y_2))$$

2. Setting the Scene (18 points)

Prove each of the following claims for arbitrary sets A, B, and C.

(a) [6 Points] $(A \setminus C) \cap (C \setminus B) = \varnothing$

(b) [12 Points] $(B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A$

3. Power Sets (16 points)

Prove or disprove the following statements:

(a) [8 Points] For any two sets S and T, it holds that:

$$\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T) \cup \mathcal{P}(S \cap T).$$

(b) [8 Points] For any two sets S and T, it holds that:

$$\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T).$$

4. Cartesian Elimination (15 points)

Let A, B, and C be non-empty sets. Prove that $(A \times B = A \times C) \rightarrow B = C$.
What happens if A is empty?

5. Modular Numerology (20 points)

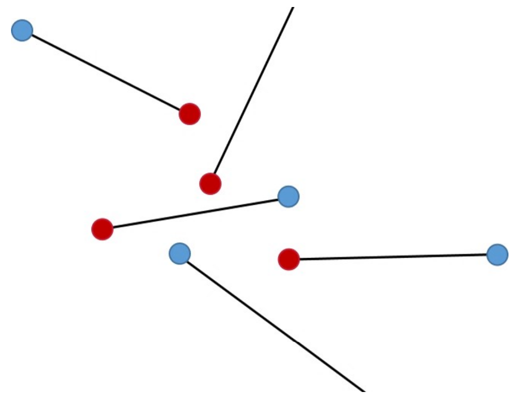
Let a, b be integers and c, m be positive integers. Prove that $a \equiv b \pmod{m}$ if and only if $ca \equiv cb \pmod{cm}$.

6. Prime Examples (15 points)

Prove that for any prime $p > 3$, either $p \equiv 1 \pmod 6$ or $p \equiv 5 \pmod 6$.

7. Extra Credit: Matchmaking (0 points)

In this problem, you will show that given n red points and n blue points in the plane such that no three points lie on a common line, it is possible to draw line segments between red-blue pairs so that all the pairs are matched and none of the line segments intersect. Assume that there are n red and n blue points fixed in the plane.



A *matching* M is a collection of n line segments connecting distinct red-blue pairs. The *total length* of a matching M is the sum of the lengths of the line segments in M. Say that a matching M is *minimal* if there is no matching with a smaller total length.

Let IsMinimal(M) be the predicate that is true precisely when M is a minimal matching. Let HasCrossing(M) be the predicate that is true precisely when there are two line segments in M that cross each other.

Give an argument in English explaining why there must be at least one matching M so that IsMinimal(M) is true, i.e.

$$\exists M \text{IsMinimal}(M))$$

Give an argument in English explaining why

$$\forall M(\text{HasCrossing}(M) \to \neg\text{IsMinimal}(M))$$

Now use the two results above to give a proof of the statement:

$$\exists M \neg\text{HasCrossing}(M).$$