CSE 311: Foundations of Computing I

Homework 2 (due Wednesday, April 11 at 11:59 PM)

Directions: *Write up carefully argued solutions to the following problems. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use results from lecture, the theorems handout, and previous homeworks without proof.*

1. Logical Equivalences (20 points)

Prove the following assertions using equivalences. You can use commutativity and associativity an arbitrary number of times in a single line of the proof.

(a) [8 Points] $p \to (q \to r) \equiv q \to (p \to r)$.

(b) [12 Points] $((p \to q) \wedge (r \to \neg q)) \to (r \to \neg p) \equiv T$.

2. Many Implications (16 points)

Our reference sheet of logical equivalences includes very little about implications (mainly just how to remove them). Boolean algebra does not include implications at all! In this problem, we will try working more directly with implications.

(a) [8 Points] Use logical equivalences to show that $(p \to q) \to p \equiv p$ and $p \to (q \to p) \equiv T$. Note that this demonstrates that $\to$ is not associative like $\vee$ and $\wedge$. (I.e., parentheses are important with $\to$.)

(b) [4 Points] Use the two equivalences just established along with $p \to p \equiv T$ and $T \to p \equiv p$ to show[1] that the following is a tautology:

$$((p \to q) \to ((p \to p) \to p)) \to (q \to p)$$

(Consider for a minute how long that would take to establish using only the reference sheet equivalences!)

(c) [4 Points] Describe how to translate any logical statement using only $\neg$, $\vee$, and $\wedge$ to an equivalent one that uses $\to$ as its only connective. Start by showing that $\neg p \equiv p \to F$. Then give equivalent forms of $p \vee q$ and $p \wedge q$.

3. Counting Courses with Combinational Logic (20 points)

In lecture 4, we considered a combinational logic example about days of class.

(a) [5 Points] Write c_1 in the product of sum form.

(b) [6 Points] Simplify the sum of product forms of c_0, c_2 using boolean algebra axioms and theorems. Make sure to cite which axioms and theorems you are using when simplifying.

(c) [9 Points] Simplify the sum of product forms of c_1 using boolean algebra axioms and theorems.

[1] Since we did not give these names, you do not need to list the names of the equivalences used *on this part only*.

4. The Bunny Hop (12 points)

Suppose that our domain of discourse is all mammals. Consider the following predicates:

- Let $R(x)$ be "x is a rabbit."
- Let $H(x)$ be "x hops."

Translate each of the following into English.

(a) [3 Points] $\exists x(R(x) \land H(x))$

(b) [3 Points] $\forall x(R(x) \to H(x))$

(c) [3 Points] $\forall x(R(x) \land H(x))$

(d) [3 Points] $\exists x(R(x) \to H(x))$

5. Book Shelf Bingo (16 points)

For this problem, let the domain of discourse be the CSE faculty and the books on their office shelves. We will also define the following predicates:

- Let Equal(x, y) be "x = y" — either the same faculty member or the same book.
- Let Same(x, y) be "x and y are books with the same content'.'
- Let OnShelf(x, y) be "book x is on the book shelf in y's office."

Translate each of the following English statements into logic using only the quantifiers $\exists$ and $\forall$ as defined in class.

(a) [4 Points] Two faculty members have books with the same content on their shelves.

(b) [4 Points] No faculty member has two books with the same content on their shelf.

(c) [4 Points] Book x is *not* the only book with its content.

(d) [4 Points] Some faculty member has a book on their shelf that is the only one (anywhere) with its content.

6. Domain of discourse (10 points)

(a) [5 Points] Give examples of predicates P and Q and a domain of discourse so that the two statements

$$\forall x(P(x) \to Q(x)) \text{ and } (\forall x P(x)) \to (\forall x Q(x))$$

are not equivalent.

(b) [5 Points] Give an example where they have the same truth value.

(c) [0 Points] **Mini extra credit:** Say that a logical connective $p \otimes q$ is *non-trivial* if it sometimes evaluates to false and sometimes evaluates to true. Is there any such connective $p \otimes q$ such that $\exists x(P(x) \otimes Q(x))$ and $\exists x P(x) \otimes \exists x Q(x)$ are logically equivalent for every domain and choice of predicates? Explain.

7. Extra credit: Comparison circuit (0 points)

In this problem, you will design a circuit with a minimal number of gates that takes a pair of length four bit strings $x_3x_2x_1x_0$ and $y_3y_2y_1y_0$ and returns a single bit indicating whether the binary integers they represent, $(x_3x_2x_1x_0)_2$ and $(y_3y_2y_1y_0)_2$, satisfy $(x_3x_2x_1x_0)_2 < (y_3y_2y_1y_0)_2$. See the following table for some examples.

$x_3x_2x_1x_0$	$y_3y_2y_1y_0$	$(x_3x_2x_1x_0)_2 < (y_3y_2y_1y_0)_2$
0101	1011	1
1100	0111	0
1101	1101	0

Design such a circuit using at most 10 AND, OR, and XOR gates. You can use an arbitrary number of NOT gates, and a single gate can have multiple inputs. (Extra credit points start at 10 gates, but if you can use fewer, you will get even more points.)