CSE 311: Foundations of Computing I

Practice Final Exam Solutions

Name:	**Sample Solutions**		
ID #:	**1234567**		
TA:	**The Best**	**Section:**	**A9**

INSTRUCTIONS:

- You have **110 minutes** to complete the exam.
- The exam is closed book. You may not use cell phones or calculators.
- All answers you want graded should be written on the exam paper.
- If you need extra space, use the back of a page. Make sure to mention that you did though.
- The problems are of varying difficulty.
- If you get stuck on a problem, move on and come back to it later.

Problem	Points	Score	Problem	Points	Score
1	15		5	15	
2	15		6	15	
3	15		7	15	
4	15		8	15	
			Σ	120	

Basic Techniques.

This part will test your ability to apply techniques that have been explicitly identified in lecture and reinforced through sections and homeworks. Remember to show your work and justify your claims.

1. Regularly Irregular [15 points]

Let $\Sigma = \{0, 1\}$. Prove that the language $L = \{x \in \Sigma^* : \#_0(x) < \#_1(x)\}$ is irregular.

Solution: Let D be an arbitrary DFA. Consider $S = \{0^n : n \geq 0\}$. Since S is infinite and D has finitely many states, we know $0^i \in S$ and $0^j \in S$ both end in the same state for some $i < j$. Append 1^j to both strings to get:

$a = 0^i 1^j$ Note that $a \in L$, because $i < j$ and $0^i 1^j \in \Sigma^*$.

$b = 0^j 1^j$ Note that $b \notin L$, because $j \not< j$.

Since a and b both end in the same state, and that state cannot both be an accept and reject state, D cannot recognize L. Since D was arbitrary, no DFA recognizes L; so, L is irregular.

2. Recurrences, Recurrences [15 points]

Define

$$T(n) = \begin{cases} n & \text{if } n = 0, 1 \\ 4T\left(\lfloor \frac{n}{2} \rfloor\right) + n & \text{otherwise} \end{cases}$$

Prove that $T(n) \le n^3$ for $n \ge 3$.

Solution: We go by strong induction on n. Let $P(n)$ be "$T(n) \le n^3$" for $n \in \mathbb{N}$.

Base Cases. When $n = 3$, $T(3) = 4T\left(\lfloor \frac{3}{2} \rfloor\right) + 3 = 4T(1) + 3 = 7 \le 27 = 3^3$.

When $n = 4$, $T(4) = 4T\left(\left\lfloor \frac{4}{2} \right\rfloor\right) + 4 = 4T(2) + 4 = 28 \le 64 = 4^3$.

When $n = 5$, $T(5) = 4T\left(\left\lfloor \frac{5}{2} \right\rfloor\right) + 5 = 4T(2) + 5 = 29 \le 4^4$.

Induction Hypothesis. Suppose $P(3) \wedge P(4) \wedge \cdots \wedge P(k)$ for some $k \ge 5$.

Induction Step. We want to prove $P(k+1)$: Note that

$$\begin{aligned}
T(k+1) &= 4T\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right) + k + 1, && \text{because } k+1 \ge 2. \\
&\le 4\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right)^3 + k + 1, && \text{by IH.} \\
&\le 4\left(\frac{k+1}{2}\right)^3 + k + 1, && \text{by def of floor.} \\
&= 4\left(\frac{(k+1)^3}{2^3}\right) + k + 1, && \text{by algebra.} \\
&= \frac{(k+1)^3}{2} + k + 1, && \text{by algebra.} \\
&= \frac{(k+1)((k+1)^2 + 2)}{2}, && \text{by algebra.} \\
&\le \frac{(k+1)((k+1)^2 + (k+1)^2)}{2}, && \text{because } (k+1)^2 \ge 2. \\
&= (k+1)^3, && \text{by algebra}
\end{aligned}$$

Thus, since the base case and induction step hold, the $P(n)$ is true for $n \ge 3$.

3. All The Machines! [15 points]
Let $\Sigma = \{0, 1, 2\}$.
Consider $L = \{w \in \Sigma^*$: Every 1 in the string has at least one 0 before and after it$\}$.

(a) (5 points) Give a regular expression that represents A.

Solution: $(0 \cup 2)^*(0(0 \cup 1 \cup 2)^*0)^*(0 \cup 2)^*$

(b) (5 points) Give a DFA that recognizes A.

Solution: Omitted.

(c) (5 points) Give a CFG that generates A.

Solution:

$$\begin{aligned} S &\to \varepsilon \mid 0\mathbf{S} \mid 2\mathbf{S} \mid 0\mathbf{ST} \\ T &\to 1\mathbf{R}0\mathbf{S} \\ R &\to \varepsilon \mid 0\mathbf{R} \mid 1\mathbf{R} \mid 2\mathbf{R} \end{aligned}$$

4. Structural CFGs [15 points]
Consider the following CFG: $\mathbf{S} \to \varepsilon \mid \mathbf{SS} \mid \mathbf{S}1 \mid \mathbf{S}01$. Another way of writing the recursive definition of this set, Q, is as follows:

- $\varepsilon \in Q$
- If $S \in Q$, then $S1 \in Q$ and $S01 \in Q$
- If $S, T \in Q$, then $ST \in Q$.

Prove, by structural induction that if $w \in Q$, then w has at least as many 1's as 0's.
Solution: We go by structal induction on w. Let $P(w)$ be "$\#_0(w) \le \#_1(w)$" for $w \in \Sigma^*$.

Base Case. When $w = \varepsilon$, note that $\#_0(w) = 0 = \#_1(w)$. So, the claim is true.

Induction Hypothesis. Suppose $P(w), P(v)$ are true for some w, v generated by the grammar.

Induction Step 1. Note that $\#_0(w1) = \#_0(w) \le \#_1(w) + 1 = \#_1(w1)$ by IH, and $\#_0(w01) = \#_0(w) + 1 \le \#_1(w) + 1 = \#_1(w01)$ by IH.

Induction Step 2. Note that $\#_0(wv) = \#_0(w) + \#_0(v) \le \#_1(w) + \#_1(v)$ by IH.

Since the claim is true for all recursive rules, the claim is true for all strings generated by the grammar.

5. Tralse! [15 points]
For each of the following answer True or False and give a short explanation of your answer.

(a) (3 points)

True **or** False	Any subset of a regular language is also regular.

Solution: False. Consider $\{0,1\}^*$ and $\{0^n1^n : n \geq 0\}$. Note that the first is regular and the second is irregular, but the second is a subset of the first.

(b) (3 points)

True **or** False	The set of programs that loop forever on at least one input is decidable.

Solution: False. If we could solve this problem, we could solve HaltNoInput. Intuitively, a program that solves this problem would have to try all inputs, but, since the program might infinte loop on some of them, it won't be able to.

(c) (3 points)

True **or** False	If $\mathbb{R} \subseteq A$ for some set A, then A is uncountable.

Solution: True. Diagonalization would still work; alternatively, if A were countable, then we could find an surjective function between $\mathbb{N}$ and $\mathbb{R}$ by skipping all the elements in A that aren't in $\mathbb{R}$.

(d) (3 points)

True **or** False	If the domain of discourse is people, the logical statement $$\exists x\ (P(x) \to \forall y\ (x \neq y \to \neg P(y))$$ can be correctly translated as "There exists a unique person who has property P".

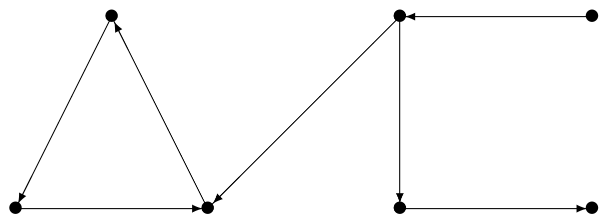
Solution: False. Any x for which $P(x)$ is false makes the entire statement true. This is not the same as there existing a unique person with property P.

(e) (3 points)

True **or** False	$\exists x\ (\forall y\ P(x,y)) \to \forall y\ (\exists x\ P(x,y))$ is true regardless of what predicate P is.

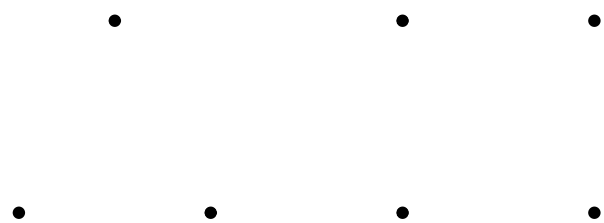
Solution: True. The left part of the implication is saying that there is a single x that works for all y; the right one is saying that for every y, we can find an x that depends on it, but the single x that works for everything will still work.

6. Relationships! [15 points]
The following is the graph of a binary relation R.



(a) (5 points) Draw the transitive-reflexive closure of R.

Solution: Omitted.



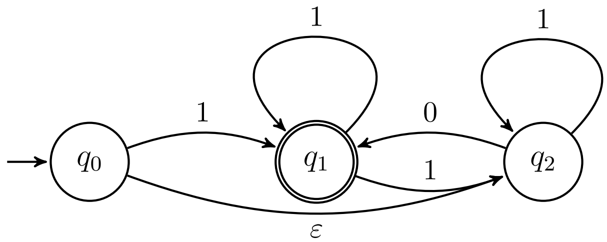
(b) (10 points) Let $S = \{(X, Y) : X, Y \in \mathcal{P}(\mathbb{N}) \wedge X \subseteq Y\}$.

Recall that R is antisymmetric iff $((a, b) \in R \wedge a \neq b) \rightarrow (b, a) \notin R$.

Prove that S is antisymmetric.

Solution: Suppose $(a, b) \in S$ and $a \neq b$. Then, by definition of S, $a \subset b$ and there is some $x \in b$ where $x \notin a$ (since they aren't equal). Then, $(b, a) \notin S$, because $b \not\subseteq a$, because $x \in b$ and $x \notin a$. So, S is antisymmetric.

7. Construction Paper! [15 points]
Convert the following NFA into a DFA using the algorithm from lecture.



Solution: Omitted.

8. Modern DFAs [15 points]
Let $\Sigma = \{0, 1, 2\}$. Construct a DFA that recognizes exactly strings with a 0 in all positions i where i mod $3 = 0$.
Solution: Omitted.