CSE 311: Foundations of Computing I

Section 9: CFGs, Relations, DFAs, NFAs, and Minimization Solutions

1. CFGs

Construct CFGs for the following languages:

(a) All binary strings that end in 00.

Solution:

$$\mathbf{S} \to 0\mathbf{S} \mid 1\mathbf{S} \mid 00$$

(b) All binary strings that contain at least three 1's.

Solution:

$$\mathbf{S} \to \mathbf{TTT}$$
$$\mathbf{T} \to 0\mathbf{T} \mid \mathbf{T}0 \mid 1\mathbf{T} \mid 1$$

(c) All binary strings with an equal number of 1's and 0's.

Solution:

$$\mathbf{S} \to 0\mathbf{S}1\mathbf{S} \mid 1\mathbf{S}0\mathbf{S} \mid \varepsilon$$

and

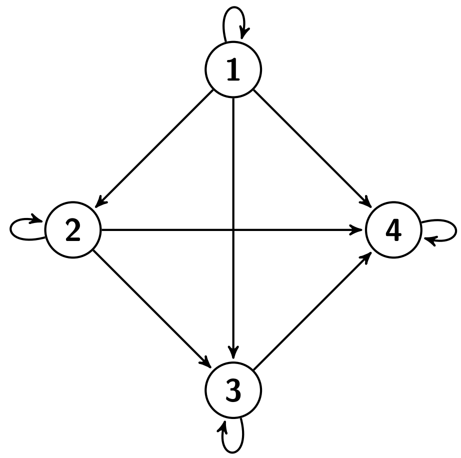
$$\mathbf{S} \to \mathbf{SS} \mid 0\mathbf{S}1 \mid 1\mathbf{S}0 \mid \varepsilon$$

both work. Note: The fact that all the strings generated have the property is easy to show (by induction) but the fact that one can generate all strings with the property is trickier. To argue this that each of these is grammars is enough one would need to consider how the difference between the # of 0's seen and the # of 1's seen occurs in prefixes of any string with the property.

2. Relations

(a) Draw the transitive-reflexive closure of $\{(1, 2), (2, 3), (3, 4)\}$.

Solution:



(b) Suppose that R is reflexive. Prove that $R \subseteq R^2$.

Solution:

Suppose $(a,b) \in R$. Since R is reflexive, we know $(b,b) \in R$ as well. Since there is a b such that $(a,b) \in R$ and $(b,b) \in R$, it follows that $(a,b) \in R^2$. Thus, $R \subseteq R^2$.

(c) Consider the relation $R = \{(x,y) \ : \ x = y+1\}$ on $\mathbb{N}$. Is R reflexive? Transitive? Symmetric? Anti-symmetric?

Solution:

It isn't reflexive, because $1 \neq 1+1$; so, $(1,1) \notin R$. It isn't symmetric, because $(2,1) \in R$ (because $2 = 1+1$), but $(1,2) \notin R$, because $1 \neq 2+1$. It isn't transitive, because note that $(3,2) \in R$ and $(2,1) \in R$, but $(3,1) \notin R$. It is anti-symmetric, because consider $(x,y) \in R$ such that $x \neq y$. Then, $x = y+1$ by definition of R. However, $(y,x) \notin R$, because $y = x - 1 \neq x + 1$.

(d) Consider the relation $S = \{(x,y) \ : \ x^2 = y^2\}$ on $\mathbb{R}$. Prove that S is reflexive, transitive, and symmetric.

Solution:

Consider $x \in \mathbb{R}$. Note that by definition of equality, $x^2 = x^2$; so, $(x,x) \in R$; so, R is reflexive.

Consider $(x,y) \in R$. Then, $x^2 = y^2$. It follows that $y^2 = x^2$; so, $(y,x) \in R$. So, R is symmetric.

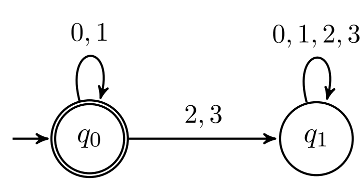
Suppose $(x,y) \in R$ and $(y,z) \in R$. Then, $x^2 = y^2$, and $y^2 = z^2$. Since equality is transitive, $x^2 = z^2$. So, $(x,z) \in R$. So, R is transitive.

3. DFAs, Stage 1

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0,1,2,3\}$.

(a) All binary strings.

Solution:

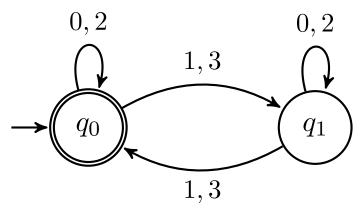


q_0**:** binary strings

q_1**:** strings that contain a character which is not 0 or 1.

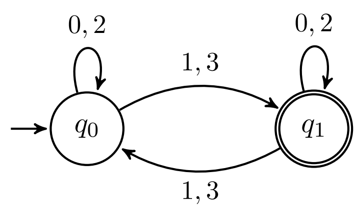
(b) All strings whose digits sum to an even number.

Solution:



(c) All strings whose digits sum to an odd number.

Solution:

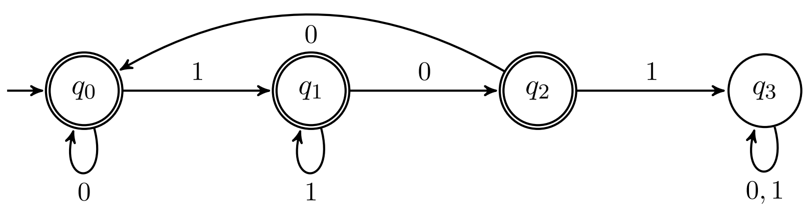


4. DFAs, Stage 2

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1\}$.

(a) All strings which do not contain the substring 101.

Solution:



q_3**:** string that contain 101.

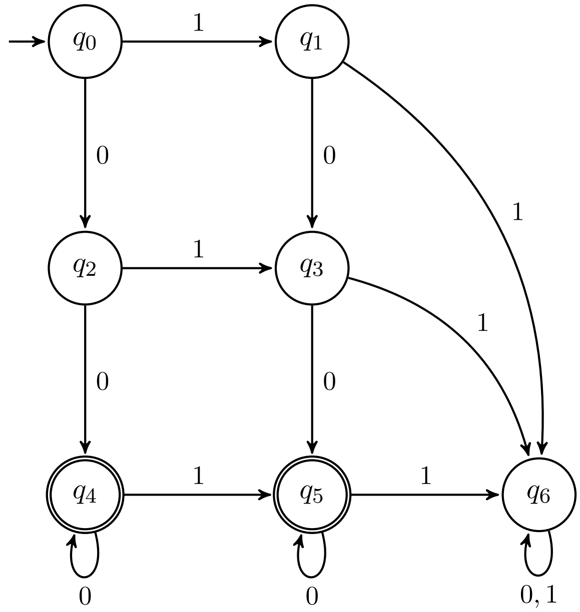
q_2**:** strings that don't contain 101 and end in 10.

q_1**:** strings that don't contain 101 and end in 1.

q_0**:** ε, 0, strings that don't contain 101 and end in 00.

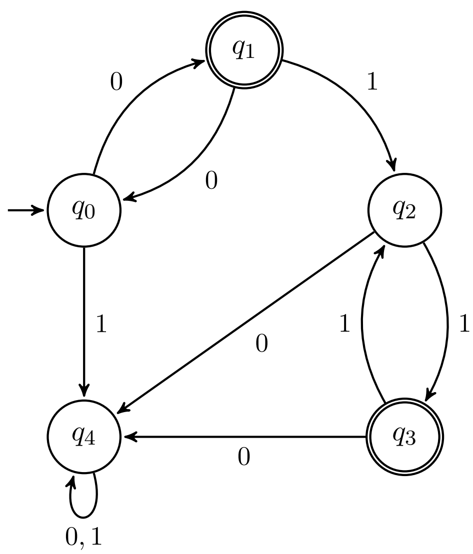
(b) All strings containing at least two 0's and at most one 1.

Solution:



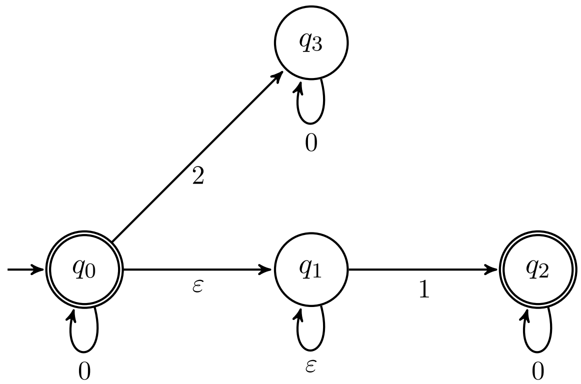
(c) All strings containing an even number of 1's and an odd number of 0's and not containing the substring 10.

Solution:



5. NFAs

(a) What language does the following NFA accept?



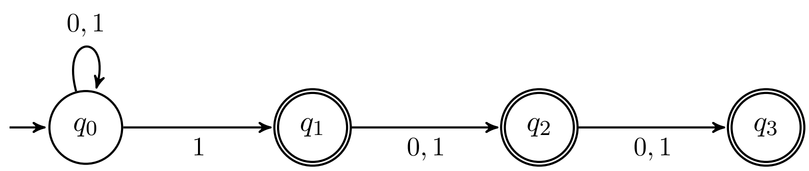
Solution:

All strings of only 0's and 1's not containing more than one 1.

(b) Create an NFA for the language "all binary strings that have a 1 as one of the last three digits".

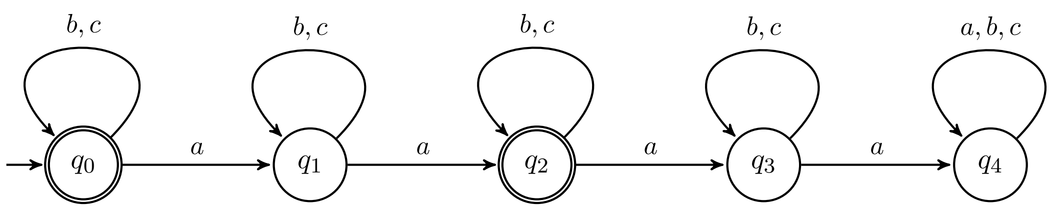
Solution:

The following is one such NFA:



6. DFA Minimization

Minimize the following DFA. For each step of the algorithm write down the groups (of states), which group was split in the step the reason for splitting that group:



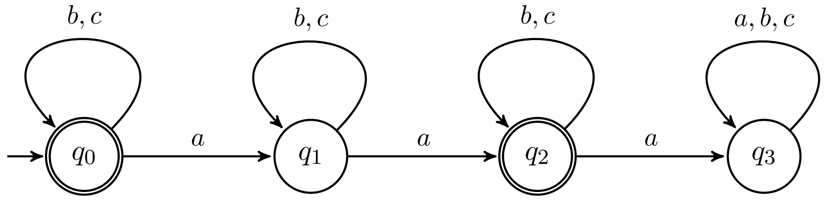
Solution:

Step 1: q_0, q_2 are final states and the rest are not final. So, we start with the initial partition with the following groups: group 1 is $\{q_0, q_2\}$ and group 2 is $\{q_1, q_3, q_4\}$.

Step 2: q_1 is sending a to group 1 while q_3, q_4 are sending a to group 2. So, we divide group 2. We get the following groups: group 1 is $\{q_0, q_2\}$, group 3 is $\{q_1\}$ and group 4 is $\{q_3, q_4\}$.

Step 3: q_0 is sending a to group 3 and q_2 is sending a to group 4. So, we divide group 1. We will have the following groups: group 3 is $\{q_1\}$, group 4 is $\{q_3, q_4\}$, group 5 is $\{q_0\}$ and group 6 is $\{q_2\}$.

The minimized DFA is the following:


b, c
b, c
b, c
a, b, c
q_0
a
q_1
a
q_2
a
q_3