1. Binary Representations
Prove that every natural number can be written as a sum of distinct powers of two. (I.e., that it has a unique binary representation.)

Solution:
Let $P(n)$ be “$n$ can written as a sum of distinct powers of two, each no larger than $n$”. We will prove $P(n)$ for all integers $n \in \mathbb{N}$ by induction.

Base Case ($n = 0$): 0 is equal to an empty sum (no powers of two), so $P(0)$ holds.

Induction Hypothesis: Assume that $P(j)$ holds for all integers $0 \leq j \leq k$ for some arbitrary $k \in \mathbb{N}$.

Induction Step: Our goal is to show $P(k + 1)$, i.e., that $k + 1$ can be written as a sum of distinct powers of two, each no larger than $k + 1$.

Let $2^\ell$ be the largest power of two not greater than $k + 1$ (i.e. $\ell = \lfloor \log_2(k + 1) \rfloor$). Let $r = (k + 1) - 2^\ell$, and note that $r < 2^\ell \leq k + 1$, so that we can apply the inductive hypothesis to $r$ to write it as a sum $r = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_t}$, where each $i_j$ is distinct and satisfies $2^{i_j} \leq r$. Note that the latter fact implies $i_j < \ell$ since $r < 2^\ell$.

Now, write $k + 1$ as $r + 2^\ell = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_t} + 2^\ell$, a sum of powers of two. Each of the $i_j$’s are distinct from each other, by assumption, and from $\ell$, since each satisfies $i_j < \ell$. Furthermore, we have $2^{i_j} \leq r < 2^\ell \leq k + 1$, so none of the powers of two in the sum are larger than $k + 1$. This shows $P(k + 1)$.

Conclusion: $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.

2. Cantelli’s Rabbits
Xavier Cantelli owns some rabbits. The number of rabbits he has in a given year is described by the function $f$:

$$
\begin{align*}
  f(0) &= 0 \\
  f(1) &= 1 \\
  f(n) &= 2f(n - 1) - f(n - 2) & \text{for } n \geq 2
\end{align*}
$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year $n$.

Solution:
Let $P(n)$ be “$f(n) = n$”. We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on $n$.

Base Cases ($n = 0, 1$): $f(0) = 0$ by definition, so $P(0)$ holds, and $f(1) = 1$, so $P(1)$ holds.

Induction Hypothesis: Assume that for some arbitrary integer $k \geq 1$, $P(j)$ holds for all $0 \leq j \leq k$. That is, for each number in this range, we have $f(j) = j$.

Induction Step: We show $P(k + 1)$, i.e. that $f(k + 1) = k + 1$. 
Since \( k + 1 \geq 2 \), we have
\[
f(k + 1) = 2f(k) - f(k - 1)
\]
\[
= 2(k) - f(k - 1)
\]
\[
= 2(k) - (k - 1)
\]
\[
= k + 1
\]
which is \( P(k + 1) \).
Therefore, \( P(n) \) is true for all \( n \in \mathbb{N} \).

3. Recursively Defined Sets of Strings

For each of the following, write a recursive definition of the sets satisfying the following properties. Briefly justify that your solution is correct.

(a) Binary strings of even length.

Solution:

Basis: \( \varepsilon \in S \).
Recursive Step: If \( x \in S \), then \( x00, x01, x10, x11 \in S \).
Exclusion Rule: Each element of \( S \) is obtained from the basis and a finite number of applications of the recursive step.

"Brief" Justification: We will show that \( x \in S \) iff \( x \) has even length (i.e., \( |x| = 2n \) for some \( n \in \mathbb{N} \)). (Note: “brief” is in quotes here. Try to write shorter explanations in your homework assignment when possible!)

Suppose \( x \in S \). If \( x \) is the empty string, then it has length 0, which is even. Otherwise, \( x \) is built up from the empty string by repeated application of the recursive step, so it is of the form \( x_1x_2\cdots x_n \), where each \( x_i \in \{00, 01, 10, 11\} \). In that case, we can see that \( |x| = |x_1| + |x_2| + \cdots + |x_n| = 2n \), which is even.

Now, suppose that \( x \) has even length. If it’s length is zero, then it is the empty string, which is in \( S \). Otherwise, it has length \( 2n \) for some \( n > 0 \), and we can write \( x \) in the form \( x_1x_2\cdots x_n \), where each \( x_i \in \{00, 01, 10, 11\} \) has length 2. Hence, we can see that \( x \) can be built up from the empty string by applying the recursive step with \( x_1 \), then \( x_2 \), and so on up to \( x_n \), which shows that \( x \in S \).

(b) Binary strings not containing 10 as a substring and having at least as many 1s as 0s.

Solution:

If the string does not contain 10, then the first 1 in the string can only be followed by more 1s. Hence, it must be of the form \( 0^m1^n \) for some \( m, n \in \mathbb{N} \). The second condition says that we have \( m \leq n \).

Basis: \( \varepsilon \in S \).
Recursive Step: If \( x \in S \), then \( 0x1 \in S \) and \( x1 \in S \).
Exclusion Rule: Each element of \( S \) is obtained from the basis and a finite number of applications of the recursive step.

Brief Justification: The empty string satisfies the property, and the recursive step cannot place a 0 after a 1 since it only adds 1s on the right. Hence, every string in \( S \) satisfies the property.

In the other direction, from our discussion above, any string of this form can be written as \( xy \), where \( x = 0^m1^n \) and \( y = 1^{n-m} \), since \( n \geq m \). We can build up the string \( x \) from the empty string by applying the rule \( x \mapsto 0x1 \) \( m \) times and then produce the string \( xy \) by applying the rule \( x \mapsto x1 \) \( n-m \) times, which shows that the string is in \( S \).