CSE 311: Foundations of Computing I

Section 7: Strong Induction and Recursive Sets Solutions

1. Binary Representations

Prove that every natural number can be written as a sum of *distinct* powers of two. (I.e., that it has a unique binary representation.)

Solution:

Let $P(n)$ be "n can written as a sum of distinct powers of two, each no larger than n". We will prove $P(n)$ for all integers $n \in \mathbb{N}$ by induction.

Base Case ($n = 0$)**:** 0 is equal to an empty sum (no powers of two), so $P(0)$ holds.

Induction Hypothesis: Assume that $P(j)$ holds for all integers $0 \leq j \leq k$ for some arbitrary $k \in \mathbb{N}$.

Induction Step: Our goal is to show $P(k+1)$. I.e., that $k+1$ can be written as a sum of distinct powers of two, each no larger than $k+1$.

Let 2^ℓ be the largest power of two not greater than $k+1$ (i.e. $\ell = \lfloor \log_2(k+1) \rfloor$). Let $r = (k+1) - 2^\ell$, and note that $r < 2^\ell \leq k+1$, so that we can apply the inductive hypothesis to r to write it as a sum $r = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_t}$, where each i_j is distinct and satisfies $2^{i_j} \leq r$. Note that the latter fact implies $i_j < \ell$ since $r < 2^\ell$.

Now, write $k+1$ as $r + 2^\ell = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_t} + 2^\ell$, a sum of powers of two. Each of the i_j's are distinct from each other, by assumption, and from ℓ, since each satisfies $i_j < \ell$. Furthermore, we have $2^{i_j} \leq r < 2^\ell \leq k+1$, so none of the powers of two in the sum are larger than $k+1$. This shows $P(k+1)$.

Conclusion: $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.

2. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in a given year is described by the function f:

$$
\begin{aligned}
f(0) &= 0 \\
f(1) &= 1 \\
f(n) &= 2f(n-1) - f(n-2) \qquad \text{for } n \geq 2
\end{aligned}
$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n.

Solution:

Let P(n) be "$f(n) = n$". We prove that P(n) is true for all $n \in \mathbb{N}$ by strong induction on n.

Base Cases ($n = 0, 1$**):** $f(0) = 0$ by definition, so P(0) holds, and $f(1) = 1$, so P(1) holds.

Induction Hypothesis: Assume that for some arbitrary integer $k \geq 1$, P(j) holds for all $0 \leq j \leq k$. That is, for each number in this range, we have $f(j) = j$.

Induction Step: We show P($k+1$), i.e. that $f(k+1) = k+1$.

Since $k+1 \geq 2$, we have

$$\begin{aligned} f(k+1) &= 2f(k) - f(k-1) && \text{Definition of } f \\ &= 2(k) - f(k-1) && \text{Inductive Hypothesis} \\ &= 2(k) - (k-1) && \text{Inductive Hypothesis} \\ &= k+1 && \text{Algebra} \end{aligned}$$

which is $\mathsf{P}(k+1)$.

Therefore, $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$.

3. Recursively Defined Sets of Strings

For each of the following, write a recursive definition of the sets satisfying the following properties. Briefly justify that your solution is correct.

(a) Binary strings of even length.

Solution:

Basis: $\varepsilon \in S$.
Recursive Step: If $x \in S$, then $x00, x01, x10, x11 \in S$.
Exclusion Rule: Each element of S is obtained from the basis and a finite number of applications of the recursive step.

"Brief" Justification: We will show that $x \in S$ iff x has even length (i.e., $|x| = 2n$ for some $n \in \mathbb{N}$). (Note: "brief" is in quotes here. Try to write shorter explanations in your homework assignment when possible!)

Suppose $x \in S$. If x is the empty string, then it has length 0, which is even. Otherwise, x is built up from the empty string by repeated application of the recursive step, so it is of the form $x_1 x_2 \cdots x_n$, where each $x_i \in \{00, 01, 10, 11\}$. In that case, we can see that $|x| = |x_1| + |x_2| + \cdots + |x_n| = 2n$, which is even.

Now, suppose that x has even length. If it's length is zero, then it is the empty string, which is in S. Otherwise, it has length $2n$ for some $n > 0$, and we can write x in the form $x_1 x_2 \cdots x_n$, where each $x_i \in \{00, 01, 10, 11\}$ has length 2. Hence, we can see that x can be built up from the empty string by applying the recursive step with x_1, then x_2, and so on up to x_n, which shows that $x \in S$.

(b) Binary strings not containing 10 as a substring and having at least as many 1s as 0s.

Solution:

If the string does not contain 10, then the first 1 in the string can only be followed by more 1s. Hence, it must be of the form $0^m 1^n$ for some $m, n \in \mathbb{N}$. The second condition says that we have $m \leq n$.

Basis: $\varepsilon \in S$.
Recursive Step: If $x \in S$, then $0x1 \in S$ and $x1 \in S$.
Exclusion Rule: Each element of S is obtained from the basis and a finite number of applications of the recursive step.

Brief Justification: The empty string satisfies the property, and the recursive step cannot place a 0 after a 1 since it only adds 1s on the right. Hence, every string in S satisfies the property.

In the other direction, from our discussion above, any string of this form can be written as xy, where $x = 0^m 1^m$ and $y = 1^{n-m}$, since $n \geq m$. We can build up the string x from the empty string by applying the rule $x \mapsto 0x1$ m times and then produce the string xy by applying the rule $x \mapsto x1$ $n-m$ times, which shows that the string is in S.