Section 6: Induction

1. Extended Euclidean Algorithm
   (a) Find the multiplicative inverse \( y \) of 7 mod 33. That is, find \( y \) such that \( 7y \equiv 1 \pmod{33} \). You should use the extended Euclidean Algorithm. Your answer should be in the range \( 0 \leq y < 33 \).

   (b) Now, solve \( 7z \equiv 2 \pmod{33} \).

2. Induction with Sums: Equality
   For any \( n \in \mathbb{N} \), define \( S_n \) to be the sum of the squares of the first \( n \) positive integers, or
   \[
   S_n = \sum_{i=1}^{n} i^2.
   \]
   For all \( n \in \mathbb{N} \), prove that \( S_n = \frac{1}{6}n(n+1)(2n+1) \).

3. A Strict Inequality
   Prove that \( 6n + 6 < 2^n \) for all \( n \geq 6 \).

4. Divisibility by Induction
   Prove that \( 9 \mid n^3 + (n+1)^3 + (n+2)^3 \) for all \( n > 1 \) by induction.

5. Another Inequality
   Prove for all \( n \in \mathbb{N} \) that, if you have numbers \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \), with \( \forall i \in [n]. a_i \leq b_i \), then:
   \[
   \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i
   \]