CSE 311: Foundations of Computing I

Section 4: English Proofs and Sets Solutions

1. Odds and Ends

Prove that for any even integer, there exists an odd integer greater than that even integer.

Solution:

Let x be an arbitrary even integer. By the definition of even, we know $x = 2y$ for some corresponding integer y. Now, we define z to be the integer $2y + 1$, which is odd by the definition of odd. By algebra, $2y + 1 > 2y$ regardless of y, so we also know $z > x$. We've now shown that there exists some integer z which is both odd and greater than x. Since x was arbitrary, we can generalize our conclusion to all even integers.

2. Primality Checking

When brute forcing if the number n is prime, you only need to check possible factors up to $\sqrt{n}$. In this problem, you'll prove why that is the case. Prove that if $n = ab$, then either a or b is at most $\sqrt{n}$.
(*Hint:* You want to prove an implication; so, start by assuming $n = ab$. Then, continue by writing out your assumption for contradiction.)

Solution:

Suppose that $n = ab$. Suppose for contradiction that $a, b > \sqrt{n}$. It follows that $ab > \sqrt{n}\sqrt{n} = n$. We clearly can't have both $n = ab$ and $n < ab$; so, this is a contradiction. It follows that a or b is at most $\sqrt{n}$.

3. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say .

(a) $A = \{1, 2, 3, 2\}$

Solution:

3

(b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\}$

Solution:

$$\begin{aligned} B &= \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \ldots\} \\ &= \{\{\}, \{\{\}\}, \{\{\}\}, \{\{\}\}, \ldots\} \\ &= \{\varnothing, \{\varnothing\}\} \end{aligned}$$

So, there are two elements in B.

(c) $C = A \times (B \cup \{7\})$

Solution:

$C = \{1, 2, 3\} \times \{\varnothing, \{\varnothing\}, 7\} = \{(a, b) \mid a \in \{1, 2, 3\}, b \in \{\varnothing, \{\varnothing\}, 7\}\}$. It follows that there are $3 \times 3 = 9$ elements in C.

(d) $D = \varnothing$

Solution:

0.

(e) $E = \{\varnothing\}$

Solution:

1.

(f) $F = \mathcal{P}(\{\varnothing\})$

Solution:

$2^1 = 2$. The elements are $F = \{\varnothing, \{\varnothing\}\}$.

4. Set = Set

Prove the following set identities.

(a) Let the universal set be $\mathcal{U}$. Prove $\overline{\overline{X}} = X$ for any set X.

Solution:

We want to prove that $S = \overline{\overline{S}}$.

$$\begin{aligned} S &= \{x \ : \ x \in S\} & \\ &= \{x \ : \ \neg\neg(x \in S)\} & \text{[Negation]} \\ &= \{x \ : \ \neg(x \notin S)\} & \text{[Definition of } \notin\text{]} \\ &= \{x \ : \ \neg(x \in \overline{S})\} & \text{[Definition of } \overline{S}\text{]} \\ &= \{x \ : \ (x \notin \overline{S})\} & \text{[Definition of } \notin\text{]} \\ &= \{x \ : \ x \in \overline{\overline{S}}\} & \text{[Definition of } \overline{\overline{S}}\text{]} \\ &= \overline{\overline{S}} & \end{aligned}$$

It follows that $S = \overline{\overline{S}}$.

(Note that if we did not have a universal set, this whole proof would be garbage.)

(b) Prove $(A \oplus B) \oplus B = A$ for any sets A, B.

Solution:

$$\begin{aligned} (A \oplus B) \oplus B &= \{x \ : \ x \in (A \oplus B) \oplus B\} & \text{[Set Definition]} \\ &= \{x \ : \ (x \in A \oplus x \in B) \oplus (x \in B)\} & \text{[Definition of } \oplus\text{]} \\ &= \{x \ : \ x \in A \oplus (x \in B \oplus x \in B)\} & \text{[Associativity of } \oplus\text{]} \\ &= \{x \ : \ x \in A \oplus (\text{F})\} & \text{[Definition of } \oplus\text{]} \\ &= \{x \ : \ x \in A\} & \text{[Definition of } \oplus\text{]} \\ &= A & \text{[Set Definition]} \end{aligned}$$

(c) Prove $A \cup B \subseteq A \cup B \cup C$ for any sets A, B, C.

Solution:

Let x be arbitrary.

$$\begin{aligned} x \in A \cup B &\rightarrow (x \in A \cup B) \vee (x \in C) \\ &\rightarrow x \in (A \cup B) \cup C \qquad \text{[Definition of } \cup] \end{aligned}$$

Thus, since $x \in A \cup B \rightarrow x \in (A \cup B) \cup C$, it follows that $A \cup B \subseteq A \cup B \cup C$, by definition of subset.

(d) Let the universal set be $\mathcal{U}$. Prove $A \cap \overline{B} \subseteq A \setminus B$ for any sets A, B.

Solution:

Let x be arbitrary.

$$\begin{aligned} x \in A \cap \overline{B} &\rightarrow x \in A \wedge x \in \overline{B} && \text{[Definition of } \cap] \\ &\rightarrow x \in A \wedge x \notin B && \text{[Definition of } \overline{B}] \\ &\rightarrow x \in A \setminus B && \text{[Definition of } \setminus] \end{aligned}$$

Thus, since $x \in A \cap \overline{B} \rightarrow x \in A \setminus B$, it follows that $A \cap \overline{B} \subseteq A \setminus B$, by definition of subset.