1. Odds and Ends
Prove that for any even integer, there exists an odd integer greater than that even integer.

Solution:
Let $x$ be an arbitrary even integer. By the definition of even, we know $x = 2y$ for some corresponding integer $y$. Now, we define $z$ to be the integer $2y + 1$, which is odd by the definition of odd. By algebra, $2y + 1 > 2y$ regardless of $y$, so we also know $z > x$. We’ve now shown that there exists some integer $z$ which is both odd and greater than $x$. Since $x$ was arbitrary, we can generalize our conclusion to all even integers.

2. Primality Checking
When brute forcing if the number $n$ is prime, you only need to check possible factors up to $\sqrt{n}$. In this problem, you’ll prove why that is the case. Prove that if $n = ab$, then either $a$ or $b$ is at most $\sqrt{n}$.

(Hint: You want to prove an implication; so, start by assuming $n = ab$. Then, continue by writing out your assumption for contradiction.)

Solution:
Suppose that $n = ab$. Suppose for contradiction that $a, b > \sqrt{n}$. It follows that $ab > \sqrt{n} \cdot \sqrt{n} = n$. We clearly can’t have both $n = ab$ and $n < ab$; so, this is a contradiction. It follows that $a$ or $b$ is at most $\sqrt{n}$.

3. How Many Elements?
For each of these, how many elements are in the set? If the set has infinitely many elements, say .

(a) $A = \{1, 2, 3, 2\}$

Solution:
3

(b) $B = \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \ldots$ 

Solution:

\[
B = \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \ldots \\
= \{\{} \}, \{\{} \}, \{\{} \}, \{\{} \}, \ldots \\
= \varnothing, \{\varnothing \}
\]

So, there are two elements in $B$.

(c) $C = A \times (B \cup \{7\})$

Solution:
$C = \{1, 2, 3\} \times \varnothing, \{\varnothing \}, 7 = \{(a, b) \mid a \in \{1, 2, 3\}, b \in \varnothing, \{\varnothing \}, 7\}$. It follows that there are $3 \times 3 = 9$ elements in $C$.

(d) $D = \varnothing$
Solution:

0.

(e) \( E = \{ \emptyset \} \)

Solution:

1.

(f) \( F = \mathcal{P}(\{ \emptyset \}) \)

Solution:

2. \( 2^1 = 2 \). The elements are \( F = \{ \emptyset , \{ \emptyset \} \} \).

4. Set = Set
Prove the following set identities.

(a) Let the universal set be \( U \). Prove \( \overline{X} = X \) for any set \( X \).

Solution:

We want to prove that \( S = \overline{S} \).

\[
S = \{ x : x \in S \} \\
= \{ x : \neg(x \in S) \} \quad \text{[Negation]} \\
= \{ x : \neg(x \notin S) \} \quad \text{[Definition of \( \notin \)]} \\
= \{ x : x \notin \overline{S} \} \quad \text{[Definition of \( \overline{S} \)]} \\
= \{ x : (x \notin \overline{S}) \} \quad \text{[Definition of \( \notin \)]} \\
= \{ x : x \in \overline{\overline{S}} \} \quad \text{[Definition of \( \overline{\overline{S}} \)]} \\
= \overline{S} 
\]

It follows that \( S = \overline{S} \).

(Note that if we did not have a universal set, this whole proof would be garbage.)

(b) Prove \( (A \oplus B) \oplus B = A \) for any sets \( A, B \).

Solution:

\[
(A \oplus B) \oplus B = \{ x : x \in (A \oplus B) \oplus B \} \quad \text{[Set Definition]} \\
= \{ x : (x \in A \oplus x \in B) \oplus (x \in B) \} \quad \text{[Definition of \( \oplus \)]} \\
= \{ x : x \in A \oplus (x \in B \oplus x \in B) \} \quad \text{[Associativity of \( \oplus \)]} \\
= \{ x : x \in A \oplus (F) \} \quad \text{[Definition of \( \oplus \)]} \\
= \{ x : x \in A \} \quad \text{[Definition of \( \oplus \)]} \\
= A \quad \text{[Set Definition]} 
\]

(c) Prove \( A \cup B \subseteq A \cup B \cup C \) for any sets \( A, B, C \).
Solution:
Let $x$ be arbitrary.

\[
\begin{align*}
x \in A \cup B & \implies (x \in A \cup B) \lor (x \in C) \\
& \implies x \in (A \cup B) \cup C \quad \text{[Definition of } \cup]\end{align*}
\]

Thus, since $x \in A \cup B \implies x \in (A \cup B) \cup C$, it follows that $A \cup B \subseteq A \cup B \cup C$, by definition of subset.

(d) Let the universal set be $\mathcal{U}$. Prove $A \cap \overline{B} \subseteq A \setminus B$ for any sets $A, B$.

Solution:
Let $x$ be arbitrary.

\[
\begin{align*}
x \in A \cap \overline{B} & \implies x \in A \land x \in \overline{B} \quad \text{[Definition of } \cap]\end{align*}
\]

\[
\begin{align*}
& \implies x \in A \land x \notin B \quad \text{[Definition of } \overline{B}\] \\
& \implies x \in A \setminus B \quad \text{[Definition of } \setminus]\end{align*}
\]

Thus, since $x \in A \cap \overline{B} \implies x \in A \setminus B$, it follows that $A \cap \overline{B} \subseteq A \setminus B$, by definition of subset.