# CSE 311: Foundations of Computing I

## Section 2: Equivalences and Predicate Logic Solutions

## 1. Equivalences

Prove that each of the following pairs of propositional formulae are equivalent using propositional equivalences.

(a) $p \leftrightarrow q$ $\quad\quad$ $(p \wedge q) \vee (\neg p \wedge \neg q)$

**Solution:**

$$
\begin{aligned}
p \leftrightarrow q &\equiv (p \rightarrow q) \wedge (q \rightarrow p) && \text{[iff is two implications]}\\
&\equiv (\neg p \vee q) \wedge (q \rightarrow p) && \text{[Law of Implication]}\\
&\equiv (\neg p \vee q) \wedge (\neg q \vee p) && \text{[Law of Implication]}\\
&\equiv ((\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p) && \text{[Distributivity]}\\
&\equiv (\neg q \wedge (\neg p \vee q)) \vee ((\neg p \vee q) \wedge p) && \text{[Commutativity]}\\
&\equiv ((\neg q \wedge \neg p) \vee (\neg q \wedge q)) \vee ((\neg p \vee q) \wedge p) && \text{[Distributivity]}\\
&\equiv ((\neg p \wedge \neg q) \vee (\neg q \wedge q)) \vee ((\neg p \vee q) \wedge p) && \text{[Commutativity]}\\
&\equiv ((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee ((\neg p \vee q) \wedge p) && \text{[Commutativity]}\\
&\equiv ((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee (p \wedge (\neg p \vee q)) && \text{[Commutativity]}\\
&\equiv ((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee ((p \wedge \neg p) \vee (p \wedge q)) && \text{[Distributivity]}\\
&\equiv ((\neg p \wedge \neg q) \vee F) \vee ((p \wedge \neg p) \vee (p \wedge q)) && \text{[Negation]}\\
&\equiv ((\neg p \wedge \neg q) \vee F) \vee (F \vee (p \wedge q)) && \text{[Negation]}\\
&\equiv (\neg p \wedge \neg q) \vee (F \vee (p \wedge q)) && \text{[Identity]}\\
&\equiv (\neg p \wedge \neg q) \vee ((p \wedge q) \vee F) && \text{[Commutativity]}\\
&\equiv (\neg p \wedge \neg q) \vee (p \wedge q) && \text{[Identity]}\\
&\equiv (p \wedge q) \vee (\neg p \wedge \neg q) && \text{[Commutativity]}
\end{aligned}
$$

(b) $\neg p \rightarrow (q \rightarrow r)$ $\quad\quad$ $q \rightarrow (p \vee r)$

**Solution:**

$$
\begin{aligned}
\neg p \rightarrow (q \rightarrow r) &\equiv \neg\neg p \vee (q \rightarrow r) && \text{[Law of Implication]}\\
&\equiv p \vee (q \rightarrow r) && \text{[Double Negation]}\\
&\equiv p \vee (\neg q \vee r) && \text{[Law of Implication]}\\
&\equiv (p \vee \neg q) \vee r && \text{[Associativity]}\\
&\equiv (\neg q \vee p) \vee r && \text{[Commutativity]}\\
&\equiv \neg q \vee (p \vee r) && \text{[Associativity]}\\
&\equiv q \rightarrow (p \vee r) && \text{[Law of Implication]}
\end{aligned}
$$

## 2. Non-equivalence

Prove that the following pairs of propositional formulae are not equivalent by finding inputs they differ on.

(a) $p \to q$ $\qquad\qquad\qquad$ $q \to p$

**Solution:**

When $p =$ T and $q =$ F, then $p \to q \equiv$ F, but $q \to p \equiv$ T.

(b) $p \to (q \wedge r)$ $\qquad\qquad\qquad$ $(p \to q) \wedge r$

**Solution:**

When $p =$ F and $r =$ F, then $p \to (q \wedge r) \equiv$ T, but $(p \to q) \wedge r \equiv$ F.

## 3. Boolean Algebra

For each of the following parts, write the logical expression using boolean algebra operators. Then, simplify it using axioms and theorems of boolean algebra.

(a) $\neg p \vee (\neg q \vee (p \wedge q))$

**Solution:**

First, we replace $\neg, \vee$, and $\wedge$. This gives us $p' + q' + pq$; note that the parentheses are not necessary in boolean algebra, because the operations are all commutative and associative. We can use DeMorgan's laws to get the slightly simpler $(pq)' + pq$. Then, we can use commutativity to get $pq + (pq)'$ and complementarity to get $1$. (Note that this is another way of saying the formula is a tautology.)

(b) $\neg(p \vee (q \wedge p))$

**Solution:**

First, we replace $\neg, \vee$, and $\wedge$ with their corresponding boolean operators, giving us $(p + (qp))'$. Applying DeMorgan's laws once gives us $p'(qp)'$, and a second time gives us $p'(q' + p')$, which is $p'(p' + q')$ by commutativity. By absorbtion, this is simply $p'$.

## 4. Canonical Forms

Consider the boolean functions $F(A, B, C)$ and $G(A, B, C)$ specified by the following truth table:

| $A$ | $B$ | $C$ | $F(A, B, C)$ | $G(A, B, C)$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 |

(a) Write the DNF and CNF expressions for $F(A, B, C)$.

**Solution:**

**DNF:** $ABC + ABC' + A'BC + A'BC' + A'B'C'$
**CNF:** $(A' + B + C')(A' + B + C)(A + B + C')$

(b) Write the DNF and CNF expressions for $G(A, B, C)$.

**Solution:**

**DNF:** $ABC' + A'BC + A'B'C$
**CNF:** $(A' + B' + C')(A' + B + C')(A' + B + C)(A + B' + C)(A + B + C)$